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K. CHANDRASEKHARAN

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CLASSROOM NOTES

REPORT OF A CONFERENCE ON MATHEMATICAL EDUCATION IN SOUTH ASIA

held at the
Tata Institute of Fundamental Research, Bombay
on 22-28 February 1956

WITH THE FINANCIAL ASSISTANCE OF
UNESCO
THE INTERNATIONAL MATHEMATICAL UNION, THE MINISTRY OF NATURAL RESOURCES AND SCIENTIFIC RESEARCH OF THE GOVERNMENT OF INDIA, THE SIR DORABJI TATA TRUST AND THE TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

BOMBAY, 22-28 FEBRUARY 1956

REPORT

1. A Conference on Mathematical Education in South Asia was held at the Tata Institute of Fundamental Research, Bombay, on 22-28 February, 1956. The Conference was the first of its kind to be held in Asia. It was attended by about seventy five mathematicians from twenty countries: Australia, Burma, Ceylon, China, France, West Germany, Hungary, India, Indonesia, Italy, Japan, Malaya, the Netherlands, Pakistan, Poland, Singapore, Thailand, the Union of Soviet Socialist Republics, the United Kingdom, and the United States; and was presided over by Professor K. Chandrasekharan.

The Conference was organized with the financial support of Unesco, the International Mathematical Union, the Government of India in the Ministry of Natural Resources and Scientific Research, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research. The proposal for the Conference was put forward by the Tata Institute of Fundamental Research, and endorsed by the National Committee for Mathematics in India, which acted as the principal agency for executing the general plan of the Conference, and for maintaining the closest liaison between the sponsoring institutions. The Tata Institute of Fundamental Research was the principal host institution.

2. Organization. The purpose of the Conference was to discuss, with special reference to South Asia, the problems of mathematical education at all levels, and to formulate plans for its sound development. An Organizing Committee with Professor K. Chandrasekharan as Chairman, and with Professors Ram Behari, E. Bompiani, K. R. Gunjikar, C. Racine, and M. H. Stone as members, drew up the programme. Professors Bompiani and Stone were nominated to this Committee by the International Mathematical Union.

News regarding the organization of the Conference was disseminated in South Asia through the good offices of the Ministry of Natural Resources and Scientific Research of the Government of India, and the Unesco Science Co-operation office at New Delhi. Invitations to send delegates to the Conference were extended to the Governments of Burma, Ceylon, Indonesia, Malaya, Pakistan, Singapore and Thailand. Members of the International Colloquium on Zeta Functions, held at the Tata Institute of Fundamental Research on 14-21 February, 1956, were invited to participate in the Conference. Universities and research institutions in India were invited to nominate one representative each. The participation of interested mathematicians from all countries, South Asian or not, was not only welcomed, but positively encouraged. Australia, China, Poland, and the U.S.S.R., for instance, were thus represented. The Organizing Committee decided, however, that only the following participants should have the right to vote : delegates sent by the Governments of the South Asian countries, members of the National Committee for Mathematics in India, members of the Organizing Committee, and those giving invited addresses. No occasion arose, however, which called for a vote.

It was decided that the Conference should function in three tiers : (i) *invited addresses*, (ii) *working groups*, and (iii) *plenary sessions*.

3. Invited addresses and special lectures. The invited addresses, which were given by experts chosen from all over the world, either dealt with the common problems of mathematical instruction confronting most countries, or with the system of education prevalent in a specific country in Asia or in Europe, or with the methods of teaching. Criticism of educational systems was balanced by constructive suggestions for their improvement. The absence of dogmatism and the presence of a freshness of approach were alike remarkable. Each address lasted forty minutes, and was followed by a discussion.

The following addresses were given :

Professor E. Bompiani (Rome): *Mathematical instruction in Italy*

Professor T. A. A. Broadbent (London): (i) *Present-day problems in English mathematical education*

Professor T. A. A. Broadbent (London): (ii) *Typography and the teaching of mathematics*

Professor G. Choquet (Paris) : *Teaching in secondary schools and research*

Professor H. Freudenthal (Utrecht): *Initiation into geometry*

Professor A. Oppenheim (Singapore) : *The problems which face mathematicians in Singapore and the Federation of Malaya*

Professor M. H. Stone (Chicago): *Some crucial problems of mathematical instruction*

Professor M. R. Siddiqi (Peshawar) and Professor L. K. Hua (Peking) who had been invited to give addresses were unable to attend the Conference.

All the addresses (with one exception) were mimeographed and distributed, immediately after delivery, to all the participants of the Conference.

On the invitation of the Organizing Committee, the following special lectures were given :

Professor A. D. Alexandrov (Leningrad) : *On mathematical education in the U.S.S.R.* (40 minutes)

Professor G. Choquet (Paris) : *New material and a new method for teaching elementary calculations in primary schools* (30 minutes)

Professor E. Marczewski (Wroclaw) : *Information on mathematical education in Poland* (10 minutes)

Professor H. F. Tuan (Peking) : *On mathematical education in Chinese universities* (15 minutes)

The invited addresses and special lectures were given in open meetings at which the attendance was larger than the membership of the Conference.

4. WORKING GROUPS. It was early recognized by the Organizing Committee that discussions between individual mathematicians, and between groups of them, should form an important part of the activities of the Conference. To keep these discussions on a serious plane, and to make them purposeful, it was necessary to conduct them in relatively small groups, of not more than thirty each. It was also considered convenient to deal with mathematical education in three stages, undergraduate, graduate, and post-graduate, which would together cover the entire range of the curriculum. Three working groups were accordingly set up. The Organizing Committee drew up a brief questionnaire common to all three working groups. The topics for discussion were put down as follows : 1. What is the purpose of teaching at this level ? 2. What should we teach ? 3. To whom and by whom ? 4. How support and how place ? 5. How to carry out ?

The working groups had some material supplied to them to start with. As early as April 1955, about one hundred mathematicians all over India were invited to send in their views on mathematical education in India to the conveners of three panels constituted for that purpose :

Panel on post-graduate teaching and research : Professor K. Chandrasekharan (Convener), Professor Ram Behari, Professor V. Ganapathy Iyer, Professor S. Minakshisundaram, Dr. B. N. Prasad, Professor C. Racine, Dr. K. G. Ramanathan.

Panel on graduate instruction : Professor C. Racine (Convener), Professor P. L. Bhatnagar, Professor V. Ganapathy Iyer, Professor S. Minakshisundaram, Professor V. V. Narlikar, Professor B. S. Madhava Rao, Professor N. R. Sen.

Panel on undergraduate instruction : Professor K. R. Gunjikar (Convener), Professor Ram Behari, Professor H. Gupta, Professor S. Mahadevan, Professor C. Racine.

The memoranda prepared by these panels were mimeographed and distributed to the respective working groups, with a view to familiarizing them with the problems which face mathematicians in a country like India. Copies of the presidential address and Professor Stone's address were supplied to the working groups right at the start, while copies of the other addresses were made available as the Conference progressed. In this way a continual supply of material was kept up, and fresh ideas were channelled in. Discussions in the working groups involved many different points of view, expressed in many different ways, often with considerable force. As a result of the co-operative effort of all the participants, however, it was possible to arrive at unanimous recommendations.

Each group had two young mathematicians attached to it as *reporters* who kept notes of the discussions. Messrs. K. Balagangadharan, M. S. Narasimhan, P. K. Raman, V. V. Rao, C. S. Seshadri and B. V. Singbal served as reporters.

When a problem came up before a working group which required a more intensive study than was possible at a regular session of the group, it was remitted to a smaller committee which reported back to the full group. An important instance of this procedure was provided by the Committee on Mathematical Education in Schools appointed by the working group on undergraduate instruction. This committee had Professor T. A. A. Broadbent as Chairman, Professor K. R. Gunjikar as co-Chairman, and Professor Aung Hla, Mr. S. D. Manerikar, Dr. R. Naidu, Mr. Poerwadi Poerwadisastro, Mr. Rabil Sitasuwana, and Miss H. K. Wong as members.

The working groups met in closed sessions of an hour and fifteen minutes each. There were nine such sessions altogether, six of them before the first plenary session, and three thereafter.

5. Plenary sessions. The plenary sessions of the Conference were devoted to a discussion of the reports of the working groups, and of those matters which were the proper concern of the Conference as a whole, like research contracts, summer schools, textbooks, and examinations, and to the formulation of conclusions in the

shape of official resolutions. It was at the plenary sessions that the work of the three groups was reviewed, integrated, and fully fashioned. The drafting of the decisions reached at the plenary sessions was done by a Drafting Committee consisting of Professor K. Chandrasekharan (Chairman), Professor Ram Behari, Professor T. A. A. Broadbent, and Professor M. H. Stone. In this way pointless digressions and long speeches were avoided, and business was transacted with efficiency and speed.

The duration of each plenary session was an hour and fifteen minutes. There were two such sessions on 25 February, and a third and final session on 28 February.

6. PROGRAMME. The Conference opened on 22 February, 1956, at 11 a.m. with a brief address of welcome by Shri Morarji R. Desai, Chief Minister of the Government of Bombay. It was formally inaugurated by Dr. H. J. Bhabha, Director of the Tata Institute of Fundamental Research. Professor E. Bompiani, Secretary of the International Mathematical Union, made a brief speech in which he expressed the interest of the Union in the Conference. Professor K. Chandrasekharan then delivered the presidential address. The whole proceedings lasted an hour. A message from the Prime Minister, Jawaharlal Nehru, reproduced in facsimile, was given to every member of the Conference.

A detailed programme of the Conference is given separately. The President of the Conference was in the chair at all the meetings. The language of the Conference was English.

At the final plenary session on 28 February, a formal resolution embodying the conclusions reached at the Conference, and put in final form by the Drafting Committee, was read out from the chair. On the motion of Professor C. Racine and Professor Ram Behari, the resolution was passed unanimously. A second resolution proposing the constitution of a Committee for Mathematics in South Asia was moved by Professor S. Tanbunyuen, and supported by Professor Ram Behari, Dr. K. S. Gangadharan, Professor Aung Hla, Professor A. Oppenheim, Professor A. L. Shaikh and

Ir. Suhakso. The speakers expressed the belief that the Conference marked the beginning of an organized effort towards the progress of mathematical education in South Asia, and considered that the proposed Committee was necessary to continue and intensify that effort. The resolution was then passed unanimously. The Conference concluded with an expression of thanks, by the President, to all those institutions and individuals who had helped to make it a success.

It was the general feeling that the Conference had been truly international in spirit, even though the resolutions passed had a special relevance and significance for South Asia.

7. The social programme for the delegates, and for the participants, included a dinner at Juhu for the delegates on 21 February ; a reception by the Vice-Chancellor of the University of Bombay, Dr. John Matthai, on 22 February ; a special performance of classical Indian dances, Kathak and Kathakali, together with a buffet supper, for the delegates, on 23 February ; a reception by the Governor of Bombay, Dr. H. K. Mahtab, at Raj Bhavan, on 24 February ; a cruise at night round the Bombay Harbour on 24 February ; an excursion by boat to Elephanta on 26 February, with lunch and tea served on board ; and a banquet to the delegates on 28 February.

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

BOMBAY, 22-28 FEBRUARY 1956

LIST OF PARTICIPANTS

AUSTRALIA

Professor A. L. Blakers
University of Western Australia
Perth

BURMA

Professor Aung Hla
University of Rangoon

CEYLON

Dr. K. S. Gangadharan
University of Ceylon
Colombo

CHINA

Dr. S. Kung
University of Peking

Professor H. F. Tuan
University of Peking

FRANCE

Professor G. Choquet
Institut Henri Poincare
11, rue Pierre Curie
Paris V

Professor S. Mandelbrojt
College de France
Paris

WEST GERMANY

Professor M. Deuring
University of Göttingen

Professor M. Eichler
University of Münster

Dr. M. Koecher
University of Münster

Professor H. Maass
University of Heidelberg

Professor H. Petersson
University of Münster

Professor C. L. Siegel
University of Göttingen

HUNGARY

Professor Paul Turan
University Matematikai Szeminariuma
Budapest

INDIA

Professor B. B. Bagi
Karnatak University

Mr. K. Balagangadharan
Tata Institute of Fundamental Research

Professor Ram Behari
University of Delhi

Dr. H. J. Bhabha
Tata Institute of Fundamental Research

Professor K. Chandrasekharan
Tata Institute of Fundamental Research

Professor G. L. Chandratreya
University of Poona

Professor B. C. Das
Utkal University

Professor K. R. Gunjikar
University of Bombay

Professor H. Gupta
Punjab University
Hoshiarpur

Dr. V. S. Krishnan
University of Madras

Professor S. Mahadevan
Indian Mathematical Society
Madras 4

Mr. S. D. Manerikar
University of Baroda

Professor S. Minakshisundaram
Andhra University
Waltair

Professor P. H. Nagappa
University of Mysore

Dr. R. Naidu
Unesco Science Co-operation Office
New Delhi

Mr. M. S. Narasimhan
Tata Institute of Fundamental Research

Dr. B. N. Prasad
University of Allahabad

Professor C. Racine
Loyola College
Madras 31

Professor C. T. Rajagopal
Ramanujan Institute of Mathematics
Madras 7

Mr. P. K. Raman
Tata Institute of Fundamental Research

Dr. K. G. Ramanathan
Tata Institute of Fundamental Research

Professor B. S. Madhava Rao
Institute of Armament Studies
Kirkee

Mr. V. V. Rao
Tata Institute of Fundamental Research

Dr. D. K. Sen
University of Bombay

Mr. C. S. Seshadri
Tata Institute of Fundamental Research

Professor B. R. Seth
Indian Institute of Technology
Kharagpur

Dr. N. G. Shabde
University of Nagpur

Professor N. M. Shah
Baroda

Dr. U. N. Singh
Muslim University
Aligarh

Mr. B. V. Singbal
Tata Institute of Fundamental Research

Professor T. P. Srivastava
University of Bihar

Professor R. Vaidyanathaswamy
Sri Venkateswara University
Tirupathi

Professor R. K. Vaish
University of Agra

INDONESIA

Mr. Saaduddin Djambek
Teacher Training Faculty
Bandung

Drs. Moerdomo
University of Indonesia
Bandung

Mr. Poerwadi Poerwadisastro
Senior High School
Malang

Ir. Suhakso
University Gadjah-Mada
Jogjakarta

ITALY

Professor E. Bompiani
Instituto Matematico
Città Universitaria
Rome

JAPAN

Professor Y. Akizuki
Kyoto University
Kyoto

Dr. I. Satake
College of General Education
University of Tokyo

MALAYA

Professor A. Oppenheim
University of Malaya
Singapore 10

THE NETHERLANDS

Professor N. G. de Bruijn
Mathematisch Institut
der Universiteit van Amsterdam

Professor H. Freudenthal
44 Franz Schubertstraat
Utrecht

PAKISTAN

Professor S. M. Kerawala
University of Karachi

Professor A. L. Shaikh
University of Sind
Hyderabad (Sind)

POLAND

Professor E. Marczewski
University of Wroclaw

SINGAPORE

Mr. G. P. Darke
Ministry of Education

Mr. Tharam Singh
Raffles Institution

Miss H. K. Wong
Teachers' Training College

THAILAND

Professor S. Tanbunyuen
Chulalongkorn University
Bangkok

Mr. Rabil Sitasuwana
Department of Educational Technique
Ministry of Education
Bangkok

UNITED KINGDOM

Professor T. A. A. Broadbent
Royal Naval College, Greenwich
London S.E.10

Professor R. A. Rankin
The University
Glasgow 2

U.S.A.

Professor S. Bochner
Fine Hall, Princeton University
Princeton, New Jersey

Professor S. Chowla
University of Colorado
Boulder, Colorado

Dr. R. C. Gunning
Fine Hall, Princeton University
Princeton, New Jersey

Professor A. Selberg
The Institute for Advanced Study
Princeton, New Jersey

Professor M. H. Stone
Eckhart Hall
University of Chicago
Chicago 37, Illinois

U.S.S.R.

Professor A. D. Alexandrov
University of Leningrad

Professor N. M. Korobov
U.S.S.R. Academy of Sciences
Moscow

Professor A. S. Monin
U.S.S.R. Academy of Sciences
Moscow

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

BOMBAY, 22-28 FEBRUARY 1956

PROGRAMME

Wednesday, February 22, 1956

11.00 A.M.	Inauguration of the Conference by Dr. H. J. Bhabha
	Presidential address by Professor K. Chandrasekharan
3.30 P.M. — 4.10 P.M.	Invited address by Professor M. H. Stone on *Some crucial problems of mathematical instruction.*

Thursday, February 23, 1956

9.45 A.M. — 11.00 A.M.	Working Group (Post-graduate)
11.15 A.M. — 12.30 P.M.	Working Group (Graduate)
3.30 P.M. — 4.10 P.M.	Invited address by Professor G. Choquet on *Teaching in secondary schools and research.*
5.00 P.M. — 5.40 P.M.	Invited address by Professor T. A. A. Broadbent on *Present-day problems in English mathematical education.*

Friday, February 24, 1956

9.45 A.M. — 11.00 A.M.	Working Group (Undergraduate)
11.15 A.M. — 12.30 P.M.	Working Group (Undergraduate)
2.30 P.M. — 3.30 P.M.	Working Group (Graduate)
3.45 P.M. — 4.30 P.M.	Working Group (Post-graduate)

Saturday, February 25, 1956

9.45 A.M. — 11.00 A.M.	Plenary session
11.30 A.M. — 12.10 P.M.	Invited address by Professor A. Oppenheim on *The problems which face mathematicians in Singapore and the Federation of Malaya.*
3.00 P.M. — 3.40 P.M.	Invited address by Professor H. Freudenthal on *Initiation into geometry.*
4.30 P.M. — 5.45 P.M.	Plenary session

Sunday, February 26, 1956

10.00 A.M. — 10.40 A.M.	Special lecture by Professor A. D. Alexandrov on *Mathematical education in the U.S.S.R.*
11.15 A.M. — 11.45 A.M.	Special lecture by Professor G. Choquet on *New material and a new method for the teaching of elementary calculations in primary schools.*

Monday, February 27, 1956

9.45 A.M. — 11.00 A.M.	Working Group (Post-graduate)
11.15 A.M. — 12.30 P.M.	Working Group (Graduate)
3.00 P.M. — 4.15 P.M.	Working Group (Undergraduate)
5.00 P.M. — 5.40 P.M.	Invited address by Professor E. Bompiani on *Mathematical instruction in Italy*.

Tuesday, February 28, 1956

9.45 A.M. — 10.25 A.M.	Invited address by Professor T. A. A. Broadbent on *Typography and the teaching of mathematics*.
11.15 A.M. — 12.30 P.M.	Plenary session
5.30 P.M. — 5.40 P.M.	Special lecture by Professor E. Marczewski on *Mathematical education in Poland*.
5.45 P.M. — 6.00 P.M.	Special lecture by Professor H. F. Tuan on *Mathematical education in Chinese universities*.

सत्यमेव जयते

Message

I send my greetings and good wishes to the International Colloquium on Zeta Functions and the South Asian Conference on Mathematical Education which are being organised by the Tata Institute of Fundamental Research in Bombay. This Institute has been recognised by the Government of India as the national centre for advanced study and fundamental research in mathematics and it is appropriate that it should hold this colloquium and conference.

Mathematics is supposed to be a dull subject, but it is increasingly recognised that it is of high importance in scientific developments today. Indeed, mathematical research has widened the horizon of the human mind tremendously and has helped in the understanding, to some extent, of nature and the physical world. It is a vehicle today of exact scientific thought. India has had the good fortune in the past to produce some very eminent mathematicians. I hope that the conferences that are being held in Bombay will foster this intellectual activity in the higher spheres of the mind and thus help in the progress of humanity.

Jawaharlal Nehru

New Delhi,
5th February, 1956.

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

BOMBAY, 22-28 FEBRUARY 1956

PRESIDENTIAL ADDRESS

By K. CHANDRASEKHARAN

I AM keenly sensible of the great honour and responsibility that vest in the presidentship of this Conference. It is hard to find a precedent in the history of Asia for a mathematical gathering of this magnitude and importance. We are particularly grateful to the institutions which have sponsored this Conference—Unesco, the International Mathematical Union, the Ministry of Natural Resources and Scientific Research of the Government of India, the Sir Dorabji Tata Trust, and the Tata Institute of Fundamental Research. I hope that the Conference will prove worthy of their trust.

This Conference is in some sense a counterpart of the International Colloquium on Zeta Functions which ended yesterday. In organizing these two meetings in conjunction, we in the Tata Institute of Fundamental Research wish to affirm our belief in the interdependence of mathematical research and mathematical education. While our primary aim is mathematical research, we wish to do whatever we can to promote the development of a sound system of mathematical education in India.

There have been, in recent years, many conferences and congresses in this part of the world, dealing with educational problems in general, but none wholly devoted to the particular problems of mathematical education. While it is essential that the countries of South Asia should first determine for themselves what educational principles and policies they wish to follow, in the large, it is equally essential that they do not stop with generalities, but translate those policies into action in every branch of education. The purpose of

this Conference is to assist in that task, as far as mathematics is concerned. We here wish to draw on the experience of the mathematically more advanced countries in the West, and to learn from their successes and failures. While we do not wish to imitate them in every respect, we do want to emulate their example in their unremitting and insistent pursuit of mathematical research and the creative assimilation of mathematical knowledge. The tremendous edifice of mathematics, is, after all, our common inheritance. We all wish to do our bit to enlarge it, to enrich it, to make it more beautiful, and thereby to contribute our share to the building up of a civilization. It seems to me that the main problem of mathematical education consists in the transmission to the young of the intellectual excitement implicit in this endeavour, and the inculcation in them of an appreciation of its significance. In this process it is clear that research and education go hand in hand, and that no one who has not experienced the thrill of mathematical discovery in some sense, or gained a perspective of the vast expanse of mathematics at some angle, can possibly inspire the young to become good mathematicians. The problems peculiar to South Asia stem from the fact that research and education have not progressed side by side, but have been tied up in a vicious circle. An antiquated system of education is at work, which has greatly reduced the research potential, so that the motive force for reform never gets to be strong enough. We are at least fifty years behind the times; what is worse, some of us hardly notice the fact. Any enduring improvement of the system requires drastic changes at all levels, from the school to the university. But we have to begin somewhere, and it seems to me that a beginning made at the university stage will accelerate reform all the way down.

We, in India, have heard a great deal said about our universities. We have had many panegyrics on their functions, from all and sundry. To repeat them now would be to get diverted from our main business. But one thing perhaps deserves to be said here, and it is this. The destiny of any people can be fulfilled only by the putting forth of the best that they are capable of in the intellectual

sphere. The creative intellect is the master key to scientific and industrial progress. Human knowledge is not something static, given once for all, but something which grows, and gets transmuted, *with every new intellectual achievement. It is the universities* of a country that ought to be the centres of its intellectual leadership. If they neglect the function of nurturing creative talent, and of giving it the fullest opportunities for growth and fulfilment, they betray their primary responsibility. The low standards of universities in South Asia are, in my opinion, largely due to a lack of recognition of this basic fact. One hears about special programmes for athletics, for military training, for the civil service, and so on—each of which should of course find its proper place—but one rarely hears of any organized, properly directed, large-scale effort for setting up schools of study and research of an international standard. The appeal exercised on some of our students by the universities of Oxford and Cambridge, or of Paris and Göttingen, or of Harvard and Princeton, is a genuine one, not so much because they are places in which every member of the staff is a superior scientist, as because they represent an ideal in action, the ideal of intellectual aspiration and achievement.

While the universities should be the mainspring of all research, it seems to me that specialized research institutes are necessary for concentrated activity in any given science. They are especially important for us in South Asia where the chief difficulty consists in providing a strong initial momentum. Such institutes have, of necessity, to be very limited in number. Although designed to supply national needs, they should be truly international in spirit in order to be effective, for in mathematics, perhaps more than in any other science, international standards are the only acceptable ones. As far as I know, there is but one such permanently established institute in the whole of South Asia at present, devoted to doctoral and post-doctoral mathematical research. I cannot say that this is an ideal state of affairs.

While the cultivation of mathematics, either in the sense of making new discoveries, or of assimilating known theories, is an

exciting form of intellectual activity, it should not be forgotten that mathematics has amply demonstrated its utility and power in its interaction with other sciences, be they physical, social or biological. In fact, the dominant characteristic of modern scientific thought may truly be described as mathematical. The nations of South Asia which are bent on industrial and technological progress cannot afford to be mathematically stagnant. We cannot expect first-rate technology to grow up beside third-rate mathematics. The attainment of a respectable level of mathematical culture should therefore be set as an immediate goal by the countries of South Asia.

The distance that separates us from that goal is great, and the obstacles are many. But we can attain it within a reasonable period of time, say thirty years, if we quicken our pace and pursue the right lines of progress. As far as India is concerned, our difficulties stem from an inadequate recognition of the value of creative intellectual activity as a part of the national drive towards prosperity. The creative scientist has not come into his own, though perhaps the administrative scientist has. This can be rectified only by the assumption of scientific leadership by the universities. Since that will take some years to happen, I think that some immediate, though temporary, remedies should be devised.

First, it is necessary to conserve the available research talent like a precious cargo. Gifted research workers in mathematics, of established merit, require a suitable atmosphere and adequate financial support in order to continue their work. Appointments in our mushrooming colleges cannot supply them with either of these needs. Nor can the standard fellowships and studentships meet their requirements, since they involve supervision and bureaucratic control. The proper solution, in the present situation, might be a system of research contracts set up by the Government and administered, for instance, by a National Committee for Mathematics, by means of which individual mathematicians can work for limited periods in surroundings of their choice, on a project offered by them and approved by the administering authority. The approval should

be based on scientific criteria determined by a pool of referees. Such contracts could be entered into not only by individuals and the Government, but by academic institutions and the Government. They can be made to cover not only research work, but also other types of activity like the writing of advanced monographs and treatises. It is only by the opening of such direct channels of assistance from the Government to the research workers that the disintegration of mathematical talent can, at present, be prevented.

Secondly, it is necessary to increase the facilities for the training of students in advanced mathematics. Mathematical research of good quality requires special preparation, particularly when the gap between university courses and active research is as wide as it is in India. The number of advanced mathematical theories which are not studied in any Indian university far outnumber those that are touched at all. Modern algebra, and algebraic topology, are, for instance, considered as radical influences or expensive vices. Mathematics as it is taught or learnt in our universities is a ghost of the dead past. It is therefore necessary, as a first step, to set up graduate schools for advanced study, with plenty of studentships, rather than pretentious and anaemic research institutes. These schools, if they are worth the name, will automatically become centres of research. We cannot hope to reap a rich mathematical harvest without having done any sowing.

Thirdly, the courses of study need to be integrated and the demoralizing influence of our system of examinations eliminated. Both the form and the content of these examinations need drastic revision. In India this problem is connected with the fact that university examinations serve as an entrance to the prized civil service examinations, in which mathematics can hardly be distinguished from a certain form of trickery. The mathematical papers set for the civil service examinations should be modernized, so that no practical reason could exist for the continuance of the present system in the universities. Until this is done, it becomes our duty to make a special provision for those students who wish to become professional mathematicians; an alternative system of courses

and examinations should be provided for them, at least from the degree stage. Such an alternative system should ensure that one who has taken the Master's degree has a sound knowledge of at least the fundamentals of analysis, algebra, geometry, and topology, together with some of their applications. It is also essential that the class-work of students receives its just share of credit alongside their performance at a formal examination.

Fourthly, the teaching staff in colleges requires rehabilitation and reinforcement if it is properly to discharge its new responsibilities. It is my opinion that the existing staff needs more leisure, and more encouragement, and more study, to cope with the problem of keeping itself up-to-date, alert to the changing aspects of our science. I feel that it has been doing a relatively fine job of teaching the Intermediate students, and criticism really begins at the graduate level. This can mean that teachers require greater opportunities for study, without financial loss,—something which can be met by the system of contracts which I have outlined, or by summer schools. A summer school for mathematics, on a small scale, was organized in Bombay several times in the past. Our colleagues in Ceylon have proposed that this should be enlarged and adapted to the needs of teachers as well as research workers. I know that the authorities of the Tata Institute of Fundamental Research are in favour of such a project, and I hope that in a year or two it will become an established fact. While the needs of the existing staff can thus be met, the principal source for fresh recruitment of staff should be the schools for advanced study and research which I have advocated. Raw graduates, who have not learnt to appreciate the significance of mathematics, should, in any case, be prevented from turning overnight into teachers of advanced students.

Fifthly, we must recognize the urgent need for suitable textbooks. Good textbooks can be a great help to a student, especially if he is not situated in a good atmosphere. They are rather scarce in this country, and, I believe, in all of South Asia. But they have to be written ; they cannot be willed into existence. The importing,

or reprinting, of books from abroad can only be a temporary solution. It is my belief that there are quite a few mathematicians in India who can write suitable textbooks at least up to the M.A. standard. But they need an inducement to write the books, they need protection against the financial loss that might be involved in their publication, and they are entitled to a share of the profit, if profit there is, at any rate, so long as we do not have a truly socialistic society ! It is unrealistic to expect commercial publishers to come to our aid, because of the financial hazards that they foresee in such an enterprise. It seems necessary, therefore, to set up a National Textbook Committee, equipped with adequate funds, which will approach competent authors and induce them, on a contractual basis, to write textbooks for a certain remuneration. If the books are published, the authors will, in addition, receive royalties. In no case will they suffer any loss. This Committee need not necessarily be different from the Committee for Research Contracts which I have previously described. But it is important that the authority which has the power to prescribe textbooks is not the agency which launches their publication.

Finally, none of the reforms we think of at the university stage can be fully effective unless the foundations are properly laid at the school stage. Mathematical instruction in schools in South Asia, at any rate in India, has remained unchanged for decades. Srinivasa Ramanujan is perhaps the most famous victim of the inefficient and inelastic system under which we operate. It is our duty to change it. In so doing we must keep in mind the aim of free and compulsory primary education for all children which almost all countries of South Asia have set themselves. We should evolve a system which, on the one hand, does not make too severe a demand on those students who do not intend to proceed to the university, and, on the other, gives every student some basic mathematical knowledge.

It is possible that my remarks are largely inspired by the state of mathematics in India. I assume, however, that many of our problems are by no means peculiar to us. I hope that some of the

suggestions I have made may prove worthy of consideration by the Conference. Profound changes in the system of mathematical education in South Asia cannot suddenly result from a single Conference like this, nor the need for change ever disappear. But this Conference can serve as a good starting point. May it succeed in that purpose. May the countries of South Asia go forward together in their pursuit of mathematics. May it be given to us to function as an effective component of the world community of mathematicians.

Tata Institute of Fundamental Research
Bombay

SOME CRUCIAL PROBLEMS OF MATHEMATICAL INSTRUCTION

By MARSHALL H. STONE

Mathematical education is entering into a critical period. There are three major factors which operate to produce the crisis : the rapid expansion of mathematical knowledge itself; the impressive penetration of mathematical thinking into the most diverse branches of learning and technology; and the universal desire to establish mass education at the primary and even the secondary levels. Teachers of mathematics can already sense the rising pressure on them to teach more mathematics to more young men and young women, at every level of education. They cannot fail to recognize that they must deal in their future classes not only with greater numbers of students but also with a wider spread of natural mathematical ability and an expanding circle of individual interests. The situation may differ in degree, but hardly in kind, from one to another of the various regions of the world. The problems which are of special interest or urgency in South Asia all have their counterparts in Europe and the Americas. The search for solutions, even though it must be influenced by the special conditions obtaining in different areas or different countries, is one in which all mathematicians have a common vital interest. It is thus no accident that mathematicians from widely separated parts of the world are brought together in a regional conference such as this. Each of us can contribute something and each of us learn something in the discussions which take place here. Those who have a special responsibility for guiding the development of mathematical instruction in South Asia will know what to take away with them from this Conference in order to attack their special problems more effectively.

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It will be good if we can, from the beginning, recognize quite frankly the essential nature of the problems with which we have to deal. We must see that they are inseparable from the progressivism of Western culture, now in the process of spreading to the entire world. Most cultures of which we have anything like an adequate knowledge appear to have been strongly conservative and essentially inimical to rapid, radical, or extensive change. The cultures of Ancient Egypt and Ancient China, notable for their stability and their millennial vigor, offer conspicuous if extreme examples of this general observation. On the other hand, human societies do not exhibit those extraordinarily rigid patterns found in certain insect societies : in even the most conservative of them cultural change takes place slowly and in small quanta, with cumulative effects which eventually can be identified as significant. Greek culture and its derivative, modern Western culture, stand out by contrast because of their eager and systematic search for new ways of thinking and doing. This characteristic progressivism is expressed materially in the advances of modern technology, but it is essentially an intellectual phenomenon, as can be seen in the growth of scholarship and pure science in the Western world. The progressive spirit inspires the search for new knowledge and its application to human affairs of every kind ; and, in order that the application may be made systematically and on a large scale, it leads logically to the introduction of general mass education. Thus the three factors at the root of the coming crisis in mathematical education are seen to be nothing else than aspects of progressivism.

The universality of this crisis reflects the universal desire to transplant some of the most highly characteristic elements of Western culture into the other cultures of the world. Clearly, we are in the midst of a kind of cultural revolution, which began when contacts between Europe and the rest of the world became numerous and increasingly significant, after about 1500. Three continents—North America, South America, and Australia as well as the northern part of Asia are now inhabited by populations predominantly

European in culture, and in race as well if exception be made for certain portions of Central America, South America and Northern Asia. While most regions of Asia and Africa have received very small numbers of European settlers, they have been subjected with few exceptions to European political control or influence for periods of varying length, during which many features of Western culture have been accepted from within or imposed from without. In the East this political control or influence is now on the point of vanishing save in areas heavily settled by Europeans; but Asia, having noted the advantages of Western technology, is more rather than less eager to adopt large segments of Western culture. Many decisions which were taken by an independent Japan in the nineteenth century are thus being repeated, mutatis mutandis, by other Asian countries in the twentieth. In consequence science and education, as understood in the West, will be intensively cultivated during the coming decades in all parts of Asia. In particular, the crucial problems of future mathematical instruction are as vital for the Asian nations as for the other nations of the world.

It is surely easiest to direct attention first to the way in which these problems emerge at the uppermost levels of education. In the universities of the world it has become urgently necessary to extend and diversify the instruction in mathematics, without prolonging unduly the periods of study required by our future technologists, scientists, and mathematicians. At the top we must plan to produce more thoroughly trained research workers in all branches of the mathematical sciences, pure mathematics included. The greatest risk we run in our approach to these problems is that of being tempted by what I shall term "the technological fallacy" —the mistaken belief that the instruction in mathematics and the natural sciences should be aimed primarily at the satisfaction of the demands of modern technology. History suggests that a healthy technology cannot be maintained without a continual vigorous probing of Nature's varied mysteries and a deep desire to understand the intricate workings of Nature's laws. One might even say that a reliable measure of the technological vigor of a country

is given by the vigor of its mathematical research. It is thus extremely imprudent for any nation to embark upon a program of education which would neglect, hamper, or discourage scientific and mathematical research in its universities. The dangers of such a course are by no means entirely avoided by accepting the compromise which would consist in confining thorough mathematical training to those future research workers deemed to have a particularly clear need for it while offering abbreviated practical instruction to those destined to become technological specialists. This compromise would provide a convenient spring-board for gradually reducing support to pure science and at the same time would burden the faculty with an ever multiplying diversity of specialized mathematical courses of a strictly practical character. In fact one of the subtler forms of the technological fallacy is expressed in the thesis that the mathematics taught to prospective technologists should be adjusted to the special requirements of each branch of technology, corresponding courses in such subjects as algebra, statistics, and the calculus being offered specifically for architects, or engineers, or chemists, or pre-medical students, or social scientists, and so on. In my own opinion the wisest plan is to offer sound basic mathematical instruction for all and to aim at a much more effectively intergrated use of basic mathematics in the technical courses offered under the various scientific departments. The basic courses in mathematics should not dwell unduly or prematurely on mathematical and logical niceties nor should they include any very large amount of material primarily of interest for advanced pure mathematics. Since the central portions of elementary university mathematics are drawn from analytic geometry, the calculus, and modern algebra, it is not difficult to design a satisfactory nucleus of basic courses and to build around it a group of more advanced courses among which the future specialist may choose according to his needs and his desires. However, it is a good deal harder to bring about the close co-ordination of courses in the various sciences with the basic courses in mathematics. In a rather long university career, I have never seen such

co-ordination attempted, let alone achieved. It is not at all unusual for the Departments of Engineering, Physics, Chemistry, Economics, and so on to lay down certain mathematical prerequisites after quite inadequate consultation, and then to expect the Department of Mathematics to adjust its curriculum so that their students may meet these requirements without inconvenience. It is, however, altogether too rare for any of these departments to ensure thereafter that the mathematical prerequisites are properly reviewed, utilized, and supplemented in its own courses. Adequate co-ordination requires very close and uninterrupted contact between departments and complete willingness to work towards the unification of the mathematical training given to students under different departments. This is a point which, I believe, deserves special emphasis in any report on current problems of higher mathematical instruction. Another aspect of university education which must be emphasized is the obligation of the faculties to avoid unnecessarily prolonging the period of study demanded of future technologists, scientists, and mathematicians. The situation in American medicine should serve to warn us of the danger that the period of preparation may become too long. In the United States it is common for a doctor to have lived more than half of his expected life-span before he is ready for the independent practice of medicine. Indeed, the future doctor, after completing four years at the university and four more in medical school, still has to spend a year or two as a hospital intern and, in all probability, a further two years in medical service with the military establishment. It is inevitable that future science students must expect to require more mathematics in their special fields and must devote more time to acquiring the necessary mathematical background. But it behooves us to save them time for what is really essential by eliminating whatever is unnecessary or secondary, by modernizing what we intend to teach, and by improving the effectiveness of our teaching. The most important step in the last direction is to analyze, rearrange, and recast the material which is found to offer the greatest difficulties to our students, until we succeed in

presenting it in the terms most suggestive to the intuition. We cannot expect a net saving of time because we must simultaneously enrich and enlarge our course offerings for undergraduates and post-graduates by the addition of new or expanded treatments of many different topics in pure and applied mathematics. What time we contrive to save for the student can be invested to broaden and deepen his mathematical understanding; and we should continually urge his teachers in other fields to save time for him too by making systematic, well-planned use of the mathematical knowledge and skill he is able to acquire in our class-rooms.

At this time I do not wish to elaborate upon the kind of curricular revision implied by the preceding remarks. Nevertheless, it may be useful for me to make a few specific observations illustrative of what I have in mind. In the United States it is a general practice to teach trigonometry and elementary algebra beyond quadratic equations to the first-year university classes, wasting a great deal of the student's time and stifling his native interest in mathematics by drilling him in dull manipulations of little eventual practical use in either pure or applied mathematics. Quite generally, in Europe as well as in America, higher algebra is still taught without benefit of the insights gained in modern approaches to the subject. Our teaching of these parts of mathematics cries out for excision, modernization and re-organization. Analytic geometry and the calculus are traditional elementary courses in which we can certainly alter some of our traditional teaching practices to great advantage. For example, in analytic geometry we fail to introduce and utilize the important vector concept, despite the simplifications and the valuable insights which it affords, because we find it difficult to teach. Not only do we thus commit a sin of omission, but we also bring it about that perforce our mathematical students first learn about vectors from clumsy and unsatisfactory treatments essayed by teachers of physics. My own experience suggests that with a little ingenuity and patience the vector concept can be taught effectively, and that the students themselves appreciate a good presentation, finding the application

of vectors to three-dimensional geometry even more rewarding than that to the plane. Needless to say, the student who acquires an elementary mastery of vectors at the beginning of his analytic geometry course can at once make effective application of them in his elementary physics course. Incidentally, I might remark that, if the discussion of trigonometry can be held off until the geometry of the circle has been discussed by vector methods, the whole subject can be vastly simplified, condensed, and illuminated. In the calculus likewise a careful analysis will disclose the advantages to be reaped by abandoning or altering some of the tradition-hallowed ways of treating the subject. If the mean-value theorem is developed early and given the central role it deserves, it becomes possible to simplify and to motivate more clearly the introduction of the definite integral and the demonstration of Taylor's theorem with remainder. By being somewhat more careful and precise in our use of terminology, we can avoid many bothersome confusions in the minds of our students. For example, the standard use of the terms "integral" and "integration" to refer to two totally distinct concepts, those of the definite integral and the indefinite integral, before any effective connection has been established between the two, virtually guarantees that the student will be confused as to the meaning and significance of the fundamental theorem of the integral calculus. It is easy enough to remedy this particular defect and others like it; but it would be a valuable contribution to our elementary teaching to discover and treat them systematically. Passing on to courses beyond the basic ones, it may be said at once that in geometry, higher albegra, and mechanics, we need to modernize and expand the material taught; that in statistics and the mathematics of computation we need to introduce new courses wherever they do not exist; and that we need to study and reorganize the whole structure of post-graduate instruction in pure mathematics with a view to ensuring that the major results of mathematical research over the last fifty years are brought within the reach of candidates for the Master's degree and within the grasp of candidates for the Doctor's degree. Those who venture to take up these tasks must

have open minds and a broad knowledge of the present state of mathematics. The tasks themselves are not too difficult; but, as we have learned at the University of Chicago, they must be worked at over a rather long period of time if difficulties and imperfections are to be eliminated. One practical point needs to be appreciated in the consideration of post-graduate mathematical instruction—namely, that under normal circumstances there can be only a limited number of university centres, even in a very large region like Western Europe or the United States, at which a fully developed post-graduate curriculum and research program can be offered. Accordingly it becomes important in guiding the development of the university system to decide where such centres should be created and where post-graduate instruction in mathematics should be limited to less ambitious goals.

What we can do in the universities has to be based on what is being or will be done in the primary and secondary schools. It is therefore essential that the schools should give adequate preparation to future university students, as well as that they should train large numbers of students who may have no desire to enter the university. The ideal of universal mass education, therefore, poses the problem of ensuring that the interests of the future university student shall not be sacrificed to the requirements of the majority or to a blind attachment to egalitarian principles. The United States provides the unfortunate example of a country which for at least twenty-five years has made this sacrifice and is just now beginning to realize, in some measure, the damage which has been done. Educationally it is clearly wrong to postpone all serious mathematical training to the university level, as some egalitarians would like us to do, because so many valuable formative years are thereby lost beyond recall. At the age of twelve or fourteen the young student's aptitude for mathematics is surely as high as it will ever be, and it should be cultivated by suitable mathematical training rather than stifled by neglect. Yet it must be admitted that the provision of such training presents a practical problem difficult for the small isolated school to solve. Perhaps the solution lies in bringing pre-university students

from the small schools together in larger central schools endowed with residential facilities. In countries where students habitually leave home at an early age, this solution is the one which is likely to be adopted. The alternative would be to provide the necessary teachers even in the smaller schools, despite the increased cost per student which would have to be borne by the school system as a whole. In these remarks we have implied our belief that pre-university students should receive separate training in mathematics. Obviously they should emerge from it with the ability to analyze, attack, and solve problems of reasonable difficulty in algebra and geometry. They should have acquired some feeling for numerical and geometrical magnitudes, some facility in expressing themselves in mathematical terms, and some practice in the art of abstraction. For them mathematics should already be a general demonstrative science rather than a collection of useful rules and formulas. In short, these students should already have been helped to take the great leap from Babylonian to Greek mathematics. In this day and age it may be suggested that in addition they should already have learned the rudiments of statistical reasoning, a suggestion which involves some definite innovations in secondary school mathematics. Furthermore, they should have learned in all these subjects some of the concepts and insights recognized as important for modern mathematics. As for the majority of secondary school students, they have a much greater need than ever before of a good working knowledge of arithmetic, simple algebra, and practical geometry—perhaps also of rudimentary statistical techniques—because these subjects are being used more and more extensively in business and industry as these activities are conducted in technically advanced countries. The United States, strangely enough, again offers an unfortunate example despite the trend of technological development there. In fact an honest contemporary survey of business and industrial requirements would undoubtedly show that in the United States the general student is almost as badly neglected in terms of his mathematical preparation for a career in business or industry as is the pre-university student for his higher studies in mathematics

and the sciences. The Asian countries should certainly not allow themselves to be misled into taking the American school as a paragon or into accepting uncritically the precepts of American educationists with regard to secondary education. Even if general mathematics in the primary and secondary schools is launched for practical reasons in a somewhat limited initial form, provision should be made for expanding its content so that it will never lag far behind the requirements of an expanding commerce and industry. The aim should be always to teach more mathematics to more students until the point is reached where every young person will learn enough to enjoy an unrestricted choice of a career within the limitations set by his native talents. In conclusion we may mention still another aspect of secondary school training in mathematics which deserves more attention than it usually receives. This is the question of continuity of instruction. It seems to me important that the pre-university student in particular should have a continuous experience with mathematics, either pure or applied, throughout his secondary school career. The alternative, which is generally followed in the United States, is to interrupt mathematical training at certain points, with unfortunate consequences for the student who later has to resume the study of mathematics in the university.

What can be done in the secondary schools depends in turn upon what is done in the primary schools. The most significant single observation to be made about primary instruction in mathematics is that it hands over the majority of its pupils to the secondary school with an abiding distrust, even a deeply ingrained fear, of mathematics. Since most human beings take a kind of innate delight in riddles and puzzles of all sorts, the explanation of this phenomenon must lie not in the nature of mathematics so much as in the manner of its teaching. Without denying the sincerity, ingenuity or persistence of the efforts made to improve the teaching of primary school mathematics, we have to be honest with ourselves in confessing that so far we have chalked up a resounding failure. It would be the counsel of despair to urge the postponement to the secondary school level of all but the barest rudiments of mathematics,

in the hope that the problem could be more easily handled there. This step would at best merely displace the problem, but might in practice be found to aggravate it. In any case the postponement, for other reasons, would surely be even more disastrous than the postponement of the present secondary mathematics to the university. On the other hand, there are many promising innovations in the teaching of elementary mathematics, like those successfully introduced by Dr. Catharine Stern* in certain American schools, and new psychological insights into the learning process, like those due to Professor Piaget, which may inspire us with lively hopes of succeeding brilliantly where in the past we have failed. I have no doubt that the key is to be found in a better understanding of the psychology of the child and the adolescent, so difficult for the untrained or unobservant adult to grasp. With a sound knowledge of the pertinent psychological principles, corresponding teaching methods of practical value in the context of mass education can then be developed and elaborated. At the same time the content of primary instruction in mathematics needs to feel the influence of the requirements and insights of modern mathematics. The discovery of conspicuously better teaching methods would inevitably have the added advantage of opening up the possibility of teaching more and more varied mathematics even at the primary level, and the choice among these opportunities should hardly be left to psychological or educational experts ignorant of the nature and uses of modern mathematics. What is indicated in these circumstances is a cooperative study of the primary curriculum by psychologists, educationists, and mathematicians. I should like to see the International Commission on Mathematical Instruction, an organ of the International Mathematical Union, do its best to promote such a study. The most obvious suggestion to be made about the primary curriculum in mathematics is that in addition to arithmetic

* Professor G. Choquet, in a special lecture given at the Conference on February 26, 1956, described the methods of M. Cuisenaire, which are identical in principle and in most details with those developed independently by Dr. Stern. The materials and manuals for Dr. Stern's methods are also available commercially, and have been used in many private and public schools in the United States.

it should include a substantial amount of intuitive and practical geometry, a subject which now clearly suffers in many school systems because of its almost total neglect at the primary level. School work intended to stimulate, direct, and develop the child's natural geometrical interests in such a way that his grasp of spatial relations and his ability to express himself in geometrical terms are systematically built up could begin in the early years and would provide a useful basis for later secondary school work. Perhaps the detailed elaboration of such a program in primary school geometry would be a peculiarly appropriate object of cooperative study at this time.

Having discussed in broad terms the problems of mathematical instruction from the top down, we may summarize our views by retracing the argument from the bottom up. In my opinion, an ideal system of mathematical instruction would take the child at his entrance into school and give him continuous mathematical training and experience up to the point where it is appropriate and advantageous for him to terminate his mathematical studies, whether this be at the primary, secondary, or university level. At the primary level, and to some extent at the secondary, instruction should be based on the most skilfully devised pedagogical methods and should be aimed at a good intuitive and practical knowledge of arithmetic, rudimentary algebra, and geometry. This portion of the mathematical curriculum should be designed as an integral part of mass education. At the secondary level a curricular differentiation should be made between pre-university and terminal students. The pre-university student should receive continuous secondary mathematical training in algebra, geometry, and the elements of statistical reasoning, designed to give him a fairly high degree of mathematical proficiency within a circumscribed mathematical domain and to enable him to proceed rapidly with further more exacting mathematical study at the university level. In the university, mathematical instruction should be modernized, enriched, unified, and skilfully graded; but it should avoid the specialization of basic courses for the benefit of specialized groups of students.

It should culminate in the strongest possible kind of post-graduate and post-doctoral research activity.

The establishment of such a system of mathematical instruction as this in any particular country would certainly involve a good deal of adaptation to special local conditions. It would also involve the resolution of many practical difficulties, particularly those of a financial order. It may be debated whether the practical obstacles to be overcome would be greater in a country where the entire educational system has to be developed virtually *ab ovo*, or in a country where educational commitments and traditions are already firmly laid down. In any case the system I have tried to describe in outline does not exist anywhere in the world to-day, except as an ideal, and its realization would cost both time and effort in any country which might wish to make it a reality. If we mathematicians desire the development of any such ideal system, we cannot rest content with merely publishing a general sketch of the scheme we would like to see adopted. We shall have to devise our scheme in detail, making careful studies of its various component elements; we shall have to explain and justify it to our fellow educators and to the public ; we shall have to struggle with the various human and material obstacles which may be opposed to its introduction. For all this we must organize ourselves and plan effective, co-ordinated measures. We must organize ourselves to study our educational problems in detail ; we must organize ourselves to call public attention to the changes which we desire to have made ; and we must organize ourselves to carry out in an effective way the decisions designed to implement our proposals. The national mathematical and educational societies, and their international counterparts, such as the International Mathematical Union, should serve our needs in these respects. Finally, let me remind you that whatever we may do we must leave room for future progressive changes. It must not be our aim or our desire to saddle future generations with rigid systems of instruction against which they, in their turn, must rebel.

University of Chicago

TEACHING IN SECONDARY SCHOOLS AND RESEARCH

By GUSTAVE CHOQUET

Since a few years, people are gathering, everywhere in the world, in conferences like this one, and becoming aware of a slow transformation which is taking place in the world. Of course, every conference is concerned with special questions relative to some particular countries—and this one is particularly important because it concerns many millions of men and women—but I think that a part of our task is also to discover and point out some universal aims.

In fact, the problems which have led us to assemble here do not concern only mathematics; we are witnessing a revolution, the birth of a fundamental discovery: we are beginning to understand one of the essential features of man.

Machines have helped us to understand what is man; they have proved that they would assume many tasks which were formerly considered as characterizing human ability: they have assumed successively the work of muscles, locomotion, speech and memory, calculation, decision among several choices, translation of languages, etc.

Therefore man has become aware that he has in himself something that machines have not; he has become aware that, above all, he is a *creator*. He is now conscious that his human dignity consists in the fact that no bound can be assigned to him; he feels able to transcend any bound.

Man is building up himself everyday, maybe our hereditary cells are the same as those of men living millions of years ago (although we are not sure of that), but what is growing and changing

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everyday is the sum of concepts created by man. We do not know what modifications are taking place in our brain when we are acquiring a new notion, but the difficulty we feel, the efforts we have to make, tell us that our brain is rebuilding itself, restructuring itself.

So, if we agree that the dignity of man consists in his thought, we see that he is in permanent evolution and progress. This progress consists in the acquisition of new notions; if this process was stopped, man would become like a mere machine.

Let us see what such a discovery implies for education. There exists a conception of education which has deep roots everywhere; it consists in believing that there is a certain amount of things which should be taught, and that schools should be like museums where skilled custodians should retain the attention of children and show them the inheritance of the past.

I think that now the time has come to change that conception and introduce a new axiom as a basis for education. If our human dignity consists in our creative ability, it is precisely this ability which should be developed in children. In other words our teaching should be no more the teaching of a museum custodian, but that of a creator.

Every teacher should kill the old man in himself and find out that he can be a discoverer and that his pupils also can be discoverers, and in fact much more powerful than himself.

And now let us look around in our respective countries. What do we see? With rare exceptions, teaching has stiffened and our colleges have become museums. In Europe I think that the worst situation occurs in secondary schools, but universities are not free from that museum conception; the best hopes come from primary schools because teachers are in closer contact with children, and also because many studies concerning abnormal childhood have aroused interest for psychology of children, and have stressed the huge creative powers of children.

I will formulate our axiom as follows :

> " Education must be the son of growing science, and not its distant relative."

Of course, many programmes should be modified ; in our existing programmes, there are monsters and horrible cancers, grown on a dying old body. But let us not blind ourselves ; let us not think that a change of programme is sufficient to rejuvenate our teaching : we know very well that even if we could change the arteries of an old man, that would not give him youth ; it might be better, if we want to renew his vitality, to give him a new hope, a new love.

Our teaching must have its new hope ; and then monsters and cancers will disappear. Our axiom does not mean that we should merely introduce the results of modern science in our teaching : that again would amount only to a change of programme. Our axiom means that every teacher should be no more a custodian in a museum, even of a museum of modern art; it means that he should become himself a creator, even at a low stage, and try to find out and arouse creativeness in his pupils.

In other words, what should be changed essentially is the conception of the role of teachers.

Of course, I know the pessimist answers to such a requirement : a teacher is not always a creative genius, he must obey programmes ; he is burdened by minor tasks (administration, etc.) ; his salary is too low so that he must have another job ; his pupils are exceptionally dull, etc.

I do not want to minimize these difficulties ; some of them are most important, and their solution is the first duty of a State and of a good Administration.

A teacher must be free of material difficulties ; he must live a decent life and be given leisure to think and work for himself.

But these necessary requirements would amount to nothing if he had not the love of teaching, the love of the child mind, the taste to discover creative powers in children.

Now, if you ask me how to acquire that love, I must confess that there is no sure method for that. However, we might try and find out the best conditions to develop it.

You know that sometimes we are bored by our daily life; we should like to travel a bit but we have not energy enough to escape; then the best thing to do is to go to the station; then to take a ticket; once we are in the train, our blood circulates more quickly, and the taste for adventure is born.

All teachers should take their ticket for adventure; and their starting point could be the study of modern mathematics. Modern mathematics is an extraordinary new world.

To teach mathematics without knowing what is modern mathematics would be to act like a museum curator who has in his museum some precious old paintings and refuses to know that there exists a modern school of painting, thinking that everything has been said and painted in the past.

So we should advise every teacher to study modern mathematics: it has such a beauty that love will come, and then they will communicate that spark of love to their pupils.

When we are teaching a given topic of which we have already a clear idea, nothing is better than to rediscover it and find out its links with other topics. Newly discovered material is like nascent atoms of hydrogen which are eager to combine with other atoms.

And now what are the practical consequences of those considerations? What can we do to have better teachers? Let us consider only teachers for secondary schools (colleges, high schools). They are formed in universities or in "normal schools". They should be given there, as professors, very good mathematicians who have done original work in a large field; those professors should give them an account of the most important structures of modern mathematics, with very precise definitions and show them—eventually by using a bit of history—that mathematics is not a dead thing, that it is changing and growing, and that it is within the scope of everybody to influence its growing.

The last year of that teaching might be given to practical training in a high school under the guidance of an experienced professor, to the study of child psychology and problems concerning the process of learning, to the mathematical study of topics closely related to what these students will have to teach (foundations of geometry, notion of orientation, elementary algebra, statistics) and finally to a study of the use of mathematics in modern science and modern life.

But experience shows that even such a good beginning is not enough, and that teachers should be given help after their university time.

One of the duties of mathematicians should be to maintain a close contact with teachers of secondary and primary schools, by books in which they would inform them, in an adequate language, of the general trend of mathematics, of its new applications to science ; where they would study, on concrete examples, the consequences of new discoveries and new notions for elementary teaching. The contact should be kept also by lectures given at regular intervals, or concentrated in time, and given during colloquia of several weeks every year.

Pedagogic research should be encouraged ; every teacher has in his class a rich living material which enables him to make pedagogical experiments ; they could discuss these experiments several times a year, at first in small, and then in larger meetings. A monthly publication written by teachers themselves with the collaboration of mathematicians and psychologists would be very helpful.

On an international scale, conferences on education should also be organized more often. When one remains in one's country, one is not easily aware of what is wrong in it ; but it will become obvious when he discusses with colleagues of other countries.

Every scientist should be aware that he is responsible for education in his own country. As soon as there is no more a living link between teaching and research, or in other words between teaching

and universities, teaching becomes a dead thing; it stiffens, and cancers begin to grow, and of course, as a result research itself suffers. On the other hand, in every country, those who administer education at the secondary and primary levels should not be jealous of their independence, and should facilitate and encourage those connections.

' Old teachers can also have an important role to play; their age gives them power and consideration; now many of them, even among those who have been considered as very gifted, do not want to understand that mathematicians can teach something to them; as they have worked hard to improve details in their teaching, they think they have attained perfection; moreover as already their brain is stiffened they are unable to acquire new notions and may be led to think sincerely that *a fortiori* their pupils would be unable to grasp them, so that they become easily enemies of every reform. They should be explained, and convinced by experience, that children's minds easily grasp new notions, and that modern mathematical notions were not invented as a pure abstract game, but are a synthesis of older and more complicated notions, and finally make possible a great economy of thought.'

Let us now study briefly the characteristic features of modern mathematics and the way they can be used in elementary education.

What interests us here in the huge development of mathematics in these last fifty years is not so much new results, new theorems, as the synthesis which was made.

During these last decades, the effort of many mathematicians consisted in discovering and studying the fundamental structures of mathematics: equivalence relations, order relations, algebraic structure, vector spaces, topology, metric spaces, differentiable manifolds, measure spaces, etc. many of which appear already, as in a germ, in the most classical mathematical entity, the set of real numbers. The importance of these structures is due to several reasons. One of them is that the simplest of them, equivalence

relations, order relations and even group structures, seem to correspond to the structure of our brain, as it results from the psychological studies of Piaget—and that implies that we should make a more extensive use of those structures in the teaching of children, even of very young children. Another reason of their importance is that one finds them everywhere in mathematics ! A very thorough study has been made of each of these structures separately. That study has realized a great economy of thought because now when a mathematician encounters a difficulty which involves several different structures, he is often able to divide that difficulty into minor ones relating to only one particular structure. Moreover every progress made in the study of one particular structure will imply a progress in all questions where that structure comes into account.

Those fundamental structures can be compared to those machines which can manufacture a single kind of object, but are able to manufacture a large number of copies within a short time, and moreover manufacture each of them with a high degree of perfection. Young mathematicians nowadays should spend at first much time in the study of the big machines of mathematics, that is to say of the fundamental structures, but then they get their reward : the theorems that they get do not concern any more a single mathematical entity but a large class of functions, of spaces, and moreover their proofs have a character of great elegance, simplicity and economy.

The broadening of the domain covered by the theorems of modern mathematics is not only interesting because it gives more general results, but also because it provides a better knowledge of each of the mathematical entities. It can be easily understood by this simple comparison : we understand better the meaning of a word or of a sentence if we know the whole page from which it has been taken.

That broadening should be considered as a fundamental principle even in the teaching of elementary mathematics. I shall give here a few examples to make precise what I mean ;

a. A line tangent to a circle was at first defined as a line which meets that circle at only one point; but its true properties were fully understood only when it was defined as a limit of secant lines. Moreover this new and more dynamic definition would now be applied to any one, convex or not.

b. Given two points A, B in the plane, the set of points M such that $\widehat{AMB} = \pi/2$ can be better studied if at the same time we study the set of points such that $\widehat{AMB} < \pi/2$ or $\widehat{AMB} > \pi/2$.

c. The notion of continuity of a function f defined on $[0, 1]$ is better understood by children if we compare f with approximating functions of a more simple type, for instance, piecewise linear.

d. Let us consider two triangles ABC, $A'B'C'$, with sides a, $b, \ldots, c'$ and angles A, B, $\ldots$, C'. A known theorem says that:

$$(a = a';\ b = b';\ C' = C) \rightarrow c = c'.$$

But that result can be much improved and at the same time simplified if we replace it by the following theorem:

In any triangle ABC, c is a function $f(a, b, C)$ of a, b, C, and is a strictly increasing function of C for $0 \leqslant C \leqslant \pi$.

e. A new method for teaching geometry, called intuitive geometry is being developed in Belgium, Italy, Netherlands; it rests partly on the same idea. For instance they start from a simple configuration, let us say a circle or a cone, and they study the family of all its plane sections.

There is another consequence of the general idea that every mathematical concept should be studied, not alone, but within its surroundings, with its various variations; it is the implication of that idea on pedagogical methods. It has been noted by psychologists and teachers that children very often can grasp more complicated situations than simple, sketchy situations; and indeed young children are always delighted when they are offered toys or when they are told stories where some apparently complicated structures appear; series of cubes that they put one inside the other; pictures

representing a box on which is painted a smaller box, on which is painted another and so on ; stories involving arithmetic progressions, or even geometric progressions of ratio equal to 2, etc.

I made experiments with my younger children, where they were led to the concept of plane convex, and non-convex, domain. Then I asked them to draw the shortest curve between two points of a non-convex domain. They were very interested, found at last the solution, and during several days filled their copy-books with more and more complicated drawings of such domains.

I play often with them at the following game : I must think of some integer x and then perform operations starting with x such as doubling, addition, subtraction that they propose to me. I tell them the integer I get at last and they must find out x. That is exactly the solution of a linear equation; but they do not know it, and they are never tired of the game.

In the same trend of ideas, Professor Turan suggested to me in a private talk that in elementary classes, the study of linear functions which seems very dull to children—and with some reason—should be preceded by the introduction of the general idea of function, and should be treated together with the study of other functions, in particular those obtained by using $|x|$, for instance $f(x) = |x + 2| - 2|x - 1|$; the great variety of graphs obtained in this way is a great excitement for children's minds.

It was often said that mathematics is a language ; in that sense, you could say that modern mathematics is a beautiful and a very precise language ; teachers should know that language and teach it by degrees to their pupils.

Even when they use only a small part of that language (for beginners) each term should be defined properly and used adequately.

I have made a survey of some textbooks of geometry and algebra for colleges and high schools, and I was horrified to see what definitions were given sometimes of straight lines, of equality, of oriented figures, of limits. Some terms are never defined ; for example nobody

knows what is a figure, a triangle, an angle. A figure is a vague entity which means sometimes a set of points, sometimes a set of subsets, or the set of all subsets which can be obtained, starting from a given set, by un-defined operations. After such a confusion one should not be astonished that children refuse to understand mathematics. After all, for a strictly logical mind, such textbooks are not understandable, and the miracle is that some children can make something out of them.

The beginning of algebra—not to speak of the beginning of geometry—is often very confused in textbooks, and many teachers do not know exactly what must be given as a definition and what must be given as a theorem ; later on there is often confusion between the notions of commutativity and associativity. Children are very sensitive to that confusion, which could be easily avoided nowadays, as there exist rigorous and elementary treatments of the beginnings of algebra.

It would take a long time to analyse in detail the contents of programmes in various countries and, as I said before, I think that programmes have a secondary importance, and that it would be better that every teacher could choose it, within certain limits, of course.

But I should like to underline at least that existing programmes have a certain tendency to contain many topics which are not essential.

In French programmes for instance, I should be glad to remove, in elementary algebra, the lengthy discussions concerning the position of a number with respect to the roots of a second degree polynomial. The time lost in such discussions could be given to stress the study of variation of various functions.

In geometry too much purity is dangerous. Classical euclidean geometry is a very beautiful theory, but a great economy of thought could be made once we know the theorem of Pythagoras. Many properties of circles or of conic sections could then be obtained

by using the regular methods of analytic geometry. That is true *a fortiori* in 3-space, which should be studied mostly using analytic geometry and vectors.

The same remarks could be made concerning the lengthy treatment which is given sometimes to the foundations of projective geometry: it seems nowadays that a projective space P_n should be defined as the space of all 1-dimensional subspaces of euclidean space R^{n+1}.

By using old and obsolete treatments just because of usage or because they are elegant, much time and energy is lost, which could be used for more essential studies. For instance, I never saw any study of convex sets in elementary textbooks, so that students have sometimes studied in great detail very complicated properties of triangles and circles and have never been told that they were convex, although the definition and elementary study of convex sets is remarkably simple and has far-reaching consequences.

Of course, a teacher who knows what are the fundamental structures of mathematics, and has understood what is really useful for higher mathematics or for applied mathematics, will not make such mistakes as I have mentioned before. But he should carefully avoid the temptation to think that what he has learnt at an advanced age does not concern young children; children are ready to understand concepts such as equivalence relations, order relations (not necessarily total order); they should not lose opportunities to understand them in all cases where these concepts are underlying.

University of Paris

PRESENT-DAY PROBLEMS IN ENGLISH MATHEMATICAL EDUCATION

By T. A. A. BROADBENT

The main educational problem in England at the moment is one which is common to many countries. How best shall we draw up a curriculum which is to be applicable to all and yet stimulating to the specially able : can we implement the policy of equal opportunity and yet avoid the sterility of equal achievement : can we frame a scheme to meet the needs of the average pupil without introducing " restrictive practices " which may depress the intelligent to the level of the mediocre : can we cater for the mass and still train the leaders ?

This problem is made at once more difficult and less difficult by the fact that England has not now and never has had a *system* of education. (Scotland must be excluded from this assertion). I do not put this forward as meritorious, or as blameworthy—it is simply a fact, an historic fact, with certain inescapable consequences. Difficulties are increased because we have no uniform basis from which to start ; they are a little diminished because the absence of a rigid system is an aid to flexibility. Unfortunately, for me it means that I cannot hope to make the present position clear without some reference to past history.

One hundred years or so ago, before the advent of anything which could be called *national* education, we had a number of schools which, beginning as charitable foundations intended to serve the immediate locality, had developed into boarding schools, drawing their pupils from the wealthier classes all over the country, controlled by a board of governors who might be regarded as the heirs to the trustees of the original charity, closely linked with, sometimes dominated by, the Established Church. From these

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schools, the two universities, Oxford and Cambridge, drew their students. For it must be remembered that these were the only two universities in England until 1830, and the development of the new universities did not become really fruitful until about 1900. Schools which remained localised, and schools which were founded by bodies dissenting from the established church, kept a vocational character giving training to the children of the lower middle class, while the boarding schools, becoming known by a curious perversion of language as the "public" schools, sent many of their pupils on to Oxford and Cambridge, from whence they passed into the professions—the Church, the law, politics. Thus education beyond the age of 12 or 13 tended to become the privilege of the few, and took its tone from the entrance requirements of the two ancient universities. During the middle and later years of the 19th century, the number of these "public" schools increased, and in addition, many of the large town grammar schools began to retain pupils to the age of 17 or 18. Thus the entrance requirements of Oxford and Cambridge, necessarily low in standard, exercised an increasing control over the school curriculum.

In geometry, the prescription of certain parts of Euclid's *Elements* as a university entrance requirement arose largely because this was the easiest way of expressing a uniform requirement in universally understood terms, and hence the domination of Euclid in the schools stemmed to some extent from the need for a regulation which should be easily and widely understood. But however valuable Euclid may be for the young professional mathematician in the making—and many of my friends have assured me that their interest in mathematics was first aroused by the logical stimulus which Euclid can give—on the whole the study of Euclid generally amounted to rote-learning of the most repulsive and undesirable type; diagrams and proofs were committed to memory, and reproduced—more or less—from memory. In 1870, some public-school teachers founded the Association for the Reform of Geometrical Teaching, chiefly to protest against this degrading and uninstructive tyranny. Success was long in coming, for the alteration of a university

regulation is a long and tedious process, even where substantial agreement in principle exists; and in this case there was at first no measure of agreement. Cayley indeed maintained, perhaps half in jest, that if Euclid was to be replaced, it should be by the study of n-dimensional geometry; from this the novice might eventually descend to the special case of $n = 2$. But in the early years of the 20th century, the universities agreed to accept any geometrical sequence which showed some degree of logical connection.

At about the same time, a strong movement was set on foot for the development of the teaching of vocational and practical mathematics. This sort of teaching, which had been quite widespread in the early 19th century in the small grammar schools, had declined as more and more schools began to look to the older universities as the ultimate aim for their pupils. Perry, who had a genius for publicity, was perhaps the best-known figure in this movement; his pupils were not nourished on the school and university classics of Todhunter and his kin, but on Molesworth's *Pocket book of engineering formulae*, and his class room was an engineering laboratory and workshop.

Though the two movements were not originally connected, they were, almost unwittingly, converging on a single objective, the creation of a mathematical curriculum designed for the average pupil. On the one hand, Euclid might be good training for the bright boy, but was deadly for the ordinary pupil; on the other, academic mathematics was out of touch with the everyday affairs of real life. The two movements had their weaknesses. Euclid offered a reasonably logical chain of theorems, and the first result of abandoning this chain was a state of chaos: a flood of substitutes was poured on to the market, each with its own plan. What was a theorem in one system was an axiom in another, an admirable training no doubt in mathematical logic, but beyond the understanding of the average boy, particularly if in the course of his education he moved from one school to another! The Perry movement, on the other hand, emphasised the purely utilitarian aspect of mathematics to the exclusion of all other values and was thus

largely responsible for the quite mistaken estimate of the place of mathematics in education which is current in England today. Further, the reform movement in geometry also laid much stress on this aspect, with the result that mathematics came to be regarded solely as a science or as a technical skill, which it is not; indeed it was often contended that mathematics should be taught as an experimental science in a laboratory, a proposal sometimes put into practice with surprising and not entirely happy results.

Those not familiar with the English system or lack of system of education may well ask at this point, " But what was the Minister of Education doing in all this?". To answer this question, we must first remember that neither the subjects of the curriculum, nor the allocation of time to these subjects, nor the way in which they are to be taught, is laid down by the Ministry. The Ministry will coordinate and advise, it will inspect and report, through its trained Inspectorate, but it is only a partner in the work of national education, which is shared between the Ministry, the local education authorities, the managers and governors of the schools, and the teachers themselves. The Ministry has no power and no organisation to set examinations. The most important public examination, the General Certificate of Education, has a general plan established by the Ministry in consultation with representatives of the schools and universities. The detailed organisation of this examination is in the hands of a number of examination boards, originally established, mostly by the universities, round about 1900, to meet the growing demand for recognisable and authoritative certificates of performance in a public examination. There are eight such boards, loosely supervised by the Ministry; papers are set and marked by panels of examiners recruited both from the schools and from the universities. The boards have certain regional affiliations, but as a rule the school takes the examination under whichever board seems to cater most suitably for the needs and aspirations of the particular school. A school in the North will not necessarily take the examination set by the Northern Universities Joint Board; much depends on the type of school and much even on the whim of the

headmaster. These examinations have three levels in each, known as " O ", " A " and " S " respectively. The " O " or Ordinary Level is intended to be suitable for the average secondary school pupil of age about 16 ; any number of subjects may be taken, from one upwards, and a certificate will be given to show simply those subjects at which the candidate has achieved a Pass Level. The " A " or Advanced Level will be taken usually two years after the Ordinary Level, and as a rule fewer subjects will be offered by the candidate. A reasonable performance at " A " level will normally qualify a student for acceptance by a university, though it will not necessarily secure him a place therein. The " S " or Scholarship Level is an examination designed for the boys and girls of high intelligence who will specialise in one or two subjects ; here a good performance will gain for the candidate a State Scholarship or Local Education Authority award to enable him to pursue a course of higher education, usually at one of the universities.

It will be clear that the universities thus exercise considerable influence on the school curriculum, directly and indirectly, through the General Certificate of Education ; directly since these examinations serve as the door to the universities ; indirectly, since university teachers are well represented on the examining boards and on the panels which prepare and mark the examination papers. It is therefore not surprising that the General Certificate is sometimes criticised on the grounds that it is too academic, and too much influenced by university policy ; that there is some substance in this criticism is perhaps shown by the fact that a ninth examining board is being constituted to give schools the option of obtaining a General Certificate for their pupils on a course more technical and more directly utilitarian than is at present customary.

The General Certificate is a post-war development, but the examinations do not differ very much in substance from the pre-war examinations, known as those for the School Certificate, which were set by the same examining bodies. Differences are so far largely administrative and organisational, rather than in specific content. In tracing developments during the present century,

therefore, we need not distinguish between the School Certificate and the General Certificate ; the latter is merely an extension and improvement on the former.

I may now return to the two movements for reform, that which began with the idea of abolishing Euclid in the schools, and that which looked to practical mathematics as its sole purpose, the Perry movement. These, I have said, were converging, perhaps involuntarily, on a single objective, the provision of a mathematical curriculum suitable for the average pupil, which should nevertheless not handicap the brighter child. Even by 1914, some keen-sighted teachers were agreeing that this could only be achieved by treating mathematics as one subject, not as a collection of isolated topics. But round about 1930, while the average teacher was still far from taking this comprehensive view, he was, from what we may almost call an administrative standpoint, beginning to call for a new deal which may ultimately go far towards completing the work of the two earlier movements and rectifying their errors.

Under the form of examination then current for the 15 or 16 year old pupil, most of these might be expected to reach a pass standard in each of three papers, one on arithmetic, one on algebra, one on geometry. Thus a weakness in geometry, common with girls and by no means unknown among boys, would gravely prejudice the pupil's examination prospects. Many teachers, naturally if perhaps regrettably more concerned with examination results than with educational progress, saw that three mixed papers, each with a due proportion of arithmetic, algebra and geometry, would give a much better chance to pupils weak in one topic, say geometry, than three separate papers each devoted solely to one of these topics. This somewhat sordid consideration came to the support of those teachers who, on better pedagogical grounds, were beginning strongly to protest against the traditional separation of mathematics into unrelated subjects. Today it may be a little difficult to realise how complete that separation often was. Even within the last 40 years, one can find instances; algebraic methods would be prohibited in the arithmetic lesson, a problem in geometry would have

to be solved without the use of coordinates if the lesson were called "pure geometry", while if the lesson were called "coordinate geometry" the same problem might have to be solved, but this time by algebraic methods only, no help or stimulus from so-called "pure geometry" being permissible. Calculus could not be used in dynamics, kinematics must not intrude into geometry.

This is perhaps not quite so ludicrous as it sounds; for it could happen that the sharpness of the prohibition served as a guard on the concentration of the less able pupil; the novice may well need to be taught the use of one tool at a time. But the disadvantages are clear and beyond compensation: the good workman must have a large collection of tools, he must be the master of each, and, above all, he must be able to select the tool for the job. This quality of "appropriateness", so valuable to the mathematician, was pushed into the background by the separation of mathematics into a system of water-tight compartments. Moreover, watertight compartments can be very dry! The desiccated topics were apt to wither early, and leave nothing behind save an unsightly and sterile residue.

We now envisage the unified course in mathematics as the present-day ideal in the school curriculum. Mathematics, in the class-room, in the text-book, in the examination, and beyond that, in adult life, is not to be thought of as a bundle of uncorrelated topics, but as a unity, a mode of thought, a universal language. Just as the good teacher of English does not separate writing from reading, prose from verse, grammar from fluency, but sees his subject as a whole, a mode of communication of thought, an end with a variety of means, so the good teacher of mathematics will see his subject as a whole, again a mode of communication of thought through a variety of means. For the outstanding teacher, this governing principle is enough and he will plan his own course; but the average teacher will depend very much on the guidance of books. In the last ten years or so, there have been numerous attempts to supply his needs; at first, such texts were, so impatient critics maintained with some show of justice, merely chapters culled from earlier texts

on arithmetic, algebra, geometry and trigonometry, sandwiched together between the covers of a single volume and presented to the unwary as a "unified course." Today this is no longer true and we have some books, only a few as yet but still some, which see mathematics, at least in its earlier stages, as a unity.

I have suggested that this completes the efforts of the two reform movements of some fifty years ago, and this, I believe, is true, though the men who led those movements at that time, if they were still with us today, might not agree with me. The academic reformers wished to abolish Euclid as a school text, but they did not wish to deprive geometry of its paramount position in school mathematics; yet this is what has happened. Nevertheless, the present situation is a logical completion of their underlying principle, that school mathematics should be related to the needs and capacities of the pupil. This was Perry's underlying principle, too, and we can claim to be completing that movement, though we cannot accept the narrow and one-sided application of it which loomed so large in Perry's view. We do not believe that the justification of mathematics can be found entirely in the field of engineering and technology, but we do believe that the pupil who has been taught to see mathematics as a whole, as a natural and indeed inevitable language for the communication of abstract and rational thought, will find it a natural and indeed inevitable way of thinking about engineering and technological problems.

So much for the present. What of the future? Here we can see two questions. First, the unified course so far has been studied mainly in connection with the examination at "O" level, for the 16-year old; what are we doing about the later school work? Secondly, what implications does the unified course hold for the position of mathematics in the school curriculum?

As regards the first question, while some teachers are not yet convinced that the unified course is an outstanding improvement, majority opinion is in its favour and looks to an extension of its principles to the later school years. Here we should expect the

calculus to be a central theme, while to this some of our more resolute reformers would add the introduction, in small and gentle doses, of some of the elementary ideas of modern algebra which are proving so potent in research mathematics, both pure and applied. Some teachers would found the calculus on kinematical notions, others would prefer an approach through graphical work, but no one would deny that kinematical ideas must enter at an early stage and occupy a prominent position, thus holding out a hand to geometry. In geometry, the old distinction between pure geometry and coordinate geometry will disappear—it has almost disappeared now—and will be closely linked on the one hand with calculus through kinematics, on the other with algebra through determinants and matrices and linear transformations. In this way, the gap which has made itself apparent of recent years between school geometry and geometry at the universities will be closed or at least narrowed. We do not expect or intend that our Sixth-formers should tackle, for instance, the formidable volumes of Hodge and Pedoe, but we hope that when they do meet that work, they will realize that, in spite of its appearance, it does deal with geometry! Dynamics will follow naturally on the kinematical content of the calculus; this will meet with the approval of the old-fashioned people, of whom I am one, who believe that a sound grasp of dynamical principles is essential for progress in applied mathematics and mathematical physics. And apart from this, it will have two good effects. First, statics will recede into its proper place, well in the background, and may even become statics and cease to be merely a disguised form of unpleasant and unilluminating algebra and trigonometry. Secondly, the old Victorian idea that applied mathematics is an unladylike topic with which no nice girl should have any acquaintance will receive its final death-blow not before time.

Perhaps more important is the impact of the new unified course on the general problem of the place of mathematics in the school curriculum.

The new outlook may well give us confidence to meet a challenge which is likely to face us in England quite soon, a challenge which

is old enough in fact, but which may now be expected to present itself with a more determined front. Does mathematics need and deserve the large proportion of school time which is at present given to it? As far as the future mathematician, physicist or technologist is concerned, the answer is not in doubt and a strong affirmative reply would hardly be denied by any competent judge. But what of the child whose future is not likely to be along such lines? The large majority of our pupils will have no occasion in their adult careers to use anything beyond the simpler processes of arithmetic; they may wish to count their change, they may even wish to follow or to dispute the calculations of the tax collector; but no more. To them, algebra, geometry, calculus have no utilitarian appeal. Why then should they study these subjects? At one time, their devotion—their compulsory devotion—to mathematics was defended, by their teachers, on the ground that mathematics, more than any other school subject, trains the mind. I wish I could believe this! But observation of my mathematical friends and contemporaries, and some small investigation into the biographies of mathematicians have forced me to the conclusion that we cannot honestly make such a claim. Mathematics is no more—and no less—a mind training, in the general sense, than the study of the classical or the modern languages, no more—if no less—than is the study, the intelligent study, of history or geography.

What then are we to do? Are we to allow mathematics to fade into the background of the curriculum for the average child, the ordinary pupil not likely to specialize in the subject? Not, I think, if we pay a little more attention to what mathematics *is*, and perhaps a little less to what mathematics *does*. Let us remember that the aim of the school is not merely to provide vocational training, but education, not at all the same thing as vocational training. Our pupils are going to be lawyers and plumbers, fishermen and doctors, housewives and journalists; yes, but for us, the teachers, this is less important than the fact that they are all going to be units in a world civilization. They are the heirs to that civilization, in a brief moment they will be the living cells and structure

of that civilization, before transmitting it to their successors. We cannot bring the child to a full knowledge of his inheritance, but we can and we must try to make each child understand that the inheritance is there waiting for him, and further, to make sure, if we can, that on coming into his estate he will not squander it nor neglect it, but will appreciate it and perhaps even enhance it. I take this to be axiomatic; for example, I want the man who like myself has small Latin and less Greek nevertheless to be aware that much of what we do or think today is conditioned by what the Sumerians and the Greeks thought 2000 or more years ago. I want our main school education to give to our pupils the keys to the whole of modern civilization.

But this admitted, the case for mathematics is strong, overwhelmingly strong, stronger by far than when based on grounds of mere utility. As the underlying foundation of so much of our present-day science and technology, and, fully as important, as a great creative art, a universal language, a basic mode of thought, the claim that mathematics is an integral part of modern culture is one which can hardly be disputed. It may be that this claim is sometimes greeted with laughter. What, say the critics, would you seriously contend that the Lebesgue integral has as great and as deep an appeal and place in our culture as, say, Paradise Lost, or the Vatican Aphrodite. We might retort that counting heads is a poor way of estimating the value of a work of art. But there is a better answer, namely, a bold affirmation that I do believe that as many people can and do appreciate the Lebesgue integral as appreciate Paradise Lost, for by this appreciation I mean to exclude all those people who will glibly tell you that Milton is a great poet and Paradise Lost a great poem, though they have never read a single book of that epic, they never intend to read a book of it, nor would they understand a line of it if they did. No; take measures of the two fields of informed appreciation, and I am sure they will not differ by very much.

How is this to help us with our pupils? There is no simple, automatically applied recipe—and it would be a bad thing if there were.

But it is easy to see where to begin. We must begin by convincing ourselves and seeing clearly in our own minds that mathematics is deeply woven into the culture of our present-day civilization; we must make sure that we have grasped this fact, that we passionately believe it to be true, that we can illustrate it with a multiplicity of detail, from science and technology, indeed, but also from architecture and law, from history and poetry. We must see the 18th century, that century of law and order, of cool rationalism, as the immediate consequence of the Newtonian philosophy. We must see the calculus, not as the gradient of a graph, but, at least in Western Europe, as the inevitable concomitant of the change from the middle ages to the modern world, from the static view of a universe fixed and immutable in its habits to the dynamic view of a changing, developing, growing complex of phenomena.

Let the teacher see mathematics as a whole, as a unity which pervades and underlies the whole structure of our present-day life in action and in thought, let him convince himself of the rightness of this view by studying its detailed implications in every field of human endeavour to which he can gain access, and he need not fear that his pupils will dislike mathematics nor that in their adult life they will shun it and forget it.

Royal Naval College, Greenwich

THE PROBLEMS WHICH FACE MATHEMATICIANS IN SINGAPORE AND THE FEDERATION OF MALAYA

By A. OPPENHEIM

Merely to give a bald account of the problems which beset mathematicians in Malaya (both in the Federation of Malaya and in Singapore) would not be helpful. The problems I feel sure though urgent and pressing to us are not new : they are in the main, the familiar ones of insufficient staff or insufficiently trained teachers, lack of appropriate textbooks suitable for the area, too traditional a curriculum, a shortage of training centres and until recently a dearth of professional mathematicians in the university : in addition there is the crucial problem of coping with a keen and growing demand for educational facilities at all levels.

It may be wise therefore to devote some time to describing the local context, to describe, however briefly and inadequately (for I am neither geographer nor historian, neither economist nor politician) the diversity and variety of Malaya, its peoples and its languages, its governments (now in a process of rapid change) and the growth of its educational systems, for there are several : Malay vernacular schools, Tamil vernacular schools, Chinese vernacular schools (at one time in several dialects, but now in Kuo-Yu), English medium schools, various training colleges, the University of Malaya (situated in Singapore) and (very shortly to be started in this same island) Nanyang, a Chinese University.

Singapore is an island some 220 square miles in extent at the extreme tip of the Malay Peninsula, just north of the Equator. It is a British Possession which is advancing rapidly towards self-government. It has indeed its own Assembly, in the main elected,

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and a Labour Front Government which came into power in April 1955.

The Federation of Malaya consists of two settlements (formerly part of the Straits Settlements to which Singapore belonged), each under a British Resident, and nine semi-autonomous States, each under a Malay Ruler, the whole controlled by the Federal Government at the capital city, Kuala Lumpur, some 250 miles north of Singapore. In each of the nine States, there is a British Adviser and in Kuala Lumpur a British High Commissioner. In the march to self-government and independence these will disappear. The recent discussions in London show that the Chief Minister for the Federation will obtain independence for his country within the Commonwealth by August 1957.

Culturally, Malaya has been influenced for centuries by the ancient civilizations of India and of China : in more recent centuries by Islam : and still more recently the West with its modern techniques has played a dominant part.

To achieve a synthesis of these diverse influences is the task which to-day faces the leaders of the various groups in Malaya : Malays, Chinese, Tamils and others too numerous to name.

For Malaya has many groups. Its total population is about 7,000,000. Nearly 85% are Malays and Chinese : the latter rather more numerous than the Malays. There are several hundred thousand Indians (mainly Tamil) and Ceylonese. In addition Malaysian peoples (including aborigines and Indonesians) Eurasians and Europeans. At one time many men from China, from India, from Europe, came to Malaya to seek a fortune, spent some years in the country, and retired to their countries of their birth for the remainder of their days. Now this tendency is disappearing. More and more Chinese and Indians and some Europeans have made Malaya their home.

The growth of population is striking. In 1931 Singapore's population was less than half-a-million : to-day it is close to one million and a quarter. Last year there were born in this small island not less than 56,000 babies. The proportion of young to old

is probably higher in Singapore than anywhere else in the world. Obviously such points have a major relevance to the educational problems of the country, for example to the building of schools and to the supply of teachers to cope with its accelerated growth in population.

I turn next to the economy of Malaya. There are three parts to be noted : first, the subsistence economy of the rural population. These grow their own rice, catch fish, tap their own rubber trees and of course, possess coconut palms. Second, there are the banking, insurance and shipping interests of the ports, the chief being Singapore, which carry on a vast international trade. And finally, without which the others will fail, there are the great rubber plantations and tin mines of Malaya. In passing it may be noted that the man who made rubber planting a success attained his 100th birth-day last December. But rubber and tin are sold on world markets sensitive to the slightest rumour. Fluctuations in price which for some years have been spectacular influence the cost of living to a marked degree.

Centuries ago Singapore had been a large and populous centre. When Stamford Raffles arrived in 1819 he found it mainly swamp inhabited by a few fishermen and their families. Under his guidance the place expanded rapidly. Raffles belonged to that rare class, an administrator who was not only able but wise and far-seeing. He proposed an Institution of Higher Learning which would undertake an extensive study of the history, literature, languages and philosophies of the surrounding countries. For this end he made a generous contribution. His thorough going scheme has not even been fulfilled although we are approaching a realisation of his vision. For men of his quality of mind are rare so that in the years which followed although numerous schools were founded, some by missions, some by Government and some by private interests no institution such as Raffles had envisaged came into existence until another century had been passed. But of this more later.

It would take too long to describe the growth of the educational systems of the Straits Settlements (as these British Colonies were

later called) and of the various Malay States to the present position. It must suffice to indicate briefly the pre-war and post-war systems and to sketch the changes which have taken place recently with an indication of what may be expected in the future.

There are Malay vernacular schools (open to all but in practice used only by Malays), Tamil vernacular schools, Chinese vernacular schools, (moving towards the sole use of Kuo-Yu, the colloquial form of Mandarin), English medium schools. The schools are run by Government : through Director of Education, Superintendents of Education and so forth, by missions (with considerable Government aid under certain conditions) and by Committees of Management.

Malay vernacular schools are chiefly to be found in rural areas. They provide free primary education in a four year course which includes reading and writing (both in arabic and romanised Malay) arithmetic and other subjects. In some schools are found fifth and sixth years. Secondary schools are now coming into existence both in the Federation and in Singapore. Shortages occur in both accommodation and staff : some students after their sixth year go on to train as teachers. The best at 10 or 11 enter English medium schools but receive two or three years intensive English before entering the general stream.

Tamil vernacular schools, to be found on most estates in Malaya usually go to Standard III, some to Standard VII : a few cover a six year course. The number of adequately trained teachers is insufficient. It should be stressed, however, that most Indians in the towns obtain education in the English medium schools.

In villages and towns of any size throughout Malaya will be found primary vernacular schools for Chinese boys and girls : some assisted by Government, some by missions but most by local committees of management which collect fees and subscriptions for support. In the past offers of aid from the Government have not been accepted, but recently strong demands for entire Government maintenance without any form of Government control have been

made. This year the Singapore Government has opened two Chinese schools, one Primary and one Secondary.

There is usually a Lower Primary extending over four years and after an Upper Primary for two years. The growth in Malaya corresponds to the growth of education in China since the revolution of 1911. The same curriculum was followed : most of the teachers came from China. The advent of a new Government in China has influenced Chinese education in Malaya to a considerable extent.

The English medium schools are open to all. Some students know English on entrance : most are taught English by the direct method. The course, formerly Primary I and II, Standards I-IX but now Primary I-VI, Forms I-VI in the Federation, Primary I-VI, Forms II-VI in Singapore, extends over ten or eleven years after which a school certificate examination conducted by the Cambridge Syndics is taken. This external examination is modified to suit local needs.

As important as the Government schools are the Aided Schools (missions). They are controlled by school management but Government which takes the fees meets the pay-roll, pays an allowance (per head) for upkeep and is responsible for half the cost of approved new building.

Teachers in aided schools if properly qualified are paid at the same rates as those in Government. Special rates exist for missionary teachers. The aided schools play a particularly important part in the education of girls.

Mention must be made also of non-aided private schools which must comply with certain regulations.

In all schools run by Government or missions, many free places are provided. Bursaries and scholarships also exist to enable bright students to go further.

It is possible that drastic changes may come about in the school system of Singapore for multi-lingualism has been accepted by the Assembly and tri-lingualism may be introduced into the schools. The difficulties need no stressing.

From the Chinese schools many students proceeded to Universities in China, but since this avenue has been in the main blocked of recent years a new University (Nanyang) has been created in Singapore by Chinese interests. This will open some time in 1956 with a staff largely recruited from Hongkong and overseas.

The English medium schools form the channel whereby students of all races proceed to higher education in such institutions as the College of Medicine and Raffles College (prior to 1949), the University of Malaya, to overseas Universities, and, for technical studies, to the Agricultural College at Serdang and the Technical College at Kuala Lumpur.

The College of Medicine founded in 1905 obtained recognition for its medical diploma some years later as a qualification registrable throughout the British Empire. Dentistry began in 1930.

In 1928 after nine years of preparation was founded Raffles College in Singapore to teach a limited range of subjects, English, History, Geography, Physics, Chemistry, Mathematics, later Economics (1933), and Education. For the College was conceived in a narrow spirit with its main function to provide teachers for the secondary schools. Conceived with such small vision with a very small staff and inadequate buildings it could not fulfil Raffle's high aims expressed so admirably a century before.

Such in brief was the position of education pre-1941 and for a short time after 1945. The war brought great changes to Malaya. In 1945 a derelict system of education had to be restored. Large numbers of unqualified teachers were employed so that a start could be made. In addition a demand for extension of education had to be met. To realise what this meant it is enough to state that in 1947 the number of children receiving education was 250,000. Less than five years later, the number exceeded 750,000. In particular in response to public pressure the pre-war demand for a University was granted after an enquiry by a Royal Commission.

The University of Malaya began in October 1949, on the basis of the existing College of Medicine and Raffles College, with Faculties

of Arts, Science and Medicine. New departments have been added : Botany, Zoology, Parasitology, Social Medicine, Malay Studies, Chinese Literature, Philosophy. Engineering has just begun mainly for the civil branch. Law and Public Administration will soon be added. And a beginning is expected shortly in Indian (mainly Tamil) studies.

To indicate the rate of expansion it is enough to say that the University began with 600 students in October 1949 : at present the number is 1300 of whom 60% are Chinese, 12% Malays, 10% Indians. Nine hundred students have applied to sit for the University Entrance Examination for 1956. Of these over 300 want to read Medicine, a demand which cannot be satisfied even though the Faculty of Medicine is now geared to produce one hundred graduates per annum by 1959. We expect by 1959 to cater for some 2,000 students. For comparison let me mention the estimate by the Commission of 1947 that the number of University students might well grow to 2,000 or more by 1972. This will indicate how pressing is the demand for higher education and how great a burden has been placed on the resources of the country. We expect also by 1958 to have a University College near Kuala Lumpur.

Before I go on to describe what is done mathematically in Malaya I ought to state that some facilities for research existed in Malaya before 1949 in Medicine and in Biology : notably the Institute of Medical Research (1900), the Botanic Gardens (Singapore 1874), Fisheries and the Rubber Research Institute.

Another point of importance is the introduction since 1951 of post-school certificate classes in many schools despite the shortage of experienced teachers to prepare students for entrance to the University. Mention should also be made of recent arrangements to enable students from Chinese Middle Schools in Singapore with knowledge of English to sit for the University Entrance Examination.

After this introduction, sketchy despite its length and suffering I fear from many omissions, it is time to turn to the question of mathematics.

So far as the elementary work in the schools is concerned, we are gradually replacing the traditionally separate subjects of Arithmetic, Algebra and Geometry by work designed to emphasize the natural links between the different topics.

In the primary stages we aim at quick and accurate responses to arithmetical problems of the kind to be met on leaving the school. Heavy manipulative work is being cut out : and in particular work with English currency. In the secondary stages, the aim is now to attain a fusion of mathematical subjects, particularly of trigonometry and geometry with the life and experience of the student, to introduce a dynamic aspect of functions as opposed to the static formula. Hence arises a reduction in the amount of formal geometry : the time so saved being used for three dimensional work, plan, elevation, simple navigation, simple trigonometry, ideas of rates of change.

Some mechanics (chiefly in connection with science) is taught. Statistics of an elementary character is being introduced.

Two main difficulties may be mentioned here :

(i) a sufficient supply of adequately trained teachers at all levels,
(ii) the supply of appropriate textbooks.

As for (i) we must suffer for some time since the great expansion of the schools has taken many teachers into administration and numerous new teachers have had only brief training. As for (ii) a Textbook Committee, set up in 1952, has made some progress. In particular a well-known English series has been revised with the author's co-operation to suit Malayan needs. Very helpful in this respect has been the use of a limited vocabulary (in English) carefully arranged. Teaching notes have been prepared.

In the Chinese Middle Schools (comparable to the English secondary schools) the position is not satisfactory. In successive years are taught Arithmetic, Algebra, Geometry, Plane Trigonometry, Advanced Algebra, Co-ordinate Geometry. Thus we find often a teacher of geometry or a teacher of algebra and not a

teacher of Mathematics. The textbooks in Chinese are built on the syllabus : one textbook for each year. In some cases Chinese translations of English texts are used : some schools use English texts but teach in Chinese.

Steps to remedy these defects (in all schools) are being taken. In the Department of Education at the University, in the new Training Colleges for teachers in Singapore and in Kota Bharu (and soon in Penang) as well as in the two training colleges in Britain used by the Federation of Malaya, vigorous work in proper training of teachers of mathematics is being undertaken. And teachers are being trained not merely for English medium schools but also for Chinese Middle Schools.

There is in the Federation a Promotion Examination in Arithmetic from the primary schools. Certain weaknesses have been revealed, for example, inability to multiply or divide mentally, lack of understanding of shortened methods. A detailed analysis is now being undertaken of a vast mass of scripts in order to arrive at precise determinations of the defects which arise.

I turn now to the work done by Raffles College from 1928 (its inception) until 1949 (with an interruption due to the occupation of Malaya from 1942-45) and the University of Malaya.

The course in Raffles College followed traditional English lines : it was modelled in the main on the Pass Degree of London University. Students (in science) read Physics, Chemistry, Pure and Applied Mathematics. The Diploma was in three parts, an examination in each part being held at the end of each year of the course. At one time the second Diploma examination was dropped, but it was found to be advisable later to reintroduce it, an experience shared also by other institutions in other countries.

As for the courses, Algebra included exponential and logarithmic series, convergence and divergence of infinite series, determinants, theory of equations, symmetric functions, Sturm's theorem (some times), location of roots, methods, such as Newton's, Horner's, Bailey's for computation of roots. Trigonometry included de Moivre's

Theorem and applications, Euler's fundamental formula, the infinite series for the trigonometric functions (and occasionally the infinite products). A short course in Spherical Trigonometry was also given.

In plane co-ordinate geometry, the general conic and in three dimensions quadrics in standard form received attention. Occasionally it was possible to study curves and surfaces. Mention should be made of a short course in geometrical conics which was found to stimulate great interest. At times also a brief introduction to projective geometry was found possible.

In calculus work in both branches was covered to include informal work with infinite integrals, differentiation and integration of infinite series, ordinary differential equations and singular solutions (very elementary) partial differential equations.

Throughout attempts were made to illuminate one field by ideas drawn from others and to introduce wherever possible some of the history of mathematics.

These methods were also used in Applied Mathematics which followed the customary courses in England for Statics, Dynamics and Hydrostatics. Included were Newton's Laws of Motion, orbits, rigid bodies, three dimensional statics. Occasional use was made of vector methods. Some students were shown Lagrange's equations.

The staff was small, two in all, with occasional assistance from a senior mathematics master in one of the schools when a member of the staff went on leave.

Of the Library, there is little to say. Adequate for the needs of a teaching institution, inadequate for research. Indeed one Principal of Raffles College asserted that a teaching institution had no business with research. Happily such views no longer prevail : all to-day are convinced that unless some research, even of a modern nature, is carried out, teaching will inevitably lose vitality.

From 1942 to 1945 Raffles College ceased to exist. In 1946 educational work began once more under circumstances of great difficulty

familiar to any country ravaged by war. From 1947 until 1949 the problems of founding the University of Malaya and coping with the increasing demand for education engaged the minds and energies of the small staff. At the outset it was agreed that Honours courses should be instituted but, since it was felt that the schools of the country would benefit more at that time from teachers with not too much specialization and since the staff of the University in Arts and in Science was still far too small to tackle both a three-year pass degree course and a three-year Honours Degree course, we began with an Honours course of one year's duration (or in some cases two) which followed for selected students the three year pass degree course. We gave in 1949 and 1950 far too much for a one year course. We gave introductions to the Foundations of Analysis. Functions of a Complex Variable, Matrices, Vector Analysis, Higher Dynamics, Hydro-dynamics, and some work on Special Functions and Statistics. From the pass degree course topics like Spherical Trigonometry have been omitted. Vector analysis has been introduced !

Fortunately since 1950 the staffing position has changed for the better. From four in 1950, of whom two were always deeply involved in administration in a growing concern, we have increased to eight in mathematics with a reasonably wide range in interests such as Statistics, Modern Algebra, Logic. The increase in staff throughout Arts and Science in the University is such that we can now begin to institute Honours Courses of two or three years duration after Intermediate for students of the right quality. Naturally the needs of physicists, chemists, engineers will not be forgotten.

The standard of students will improve with the output of teachers from the training colleges and the university. Genuine mathematical ability exists : it needs to be fostered but the resources of Malaya are not sufficient to maintain a team of mathematicians of the highest quality, covering all branches of our vast subject, able to direct and inspire young and vigorous talent. Plainly we must expect to find places in centres abroad for our best students and

hope that from time to time distinguished mathematicians will spend a few months in Malaya.

It should be pointed out also that subjects like Medicine attract a disproportionate number of able students. The reasons are clear : prestige and great financial rewards. There is also an unhappy tendency in Government circles to believe that only those trained in the Social Sciences are fitted for the highest ranks of Government service, although perhaps experience should have shown by now that those trained to administer are not necessarily good administrators. On the Library side we are now indeed fortunate. Funds have been made freely available. Many periodicals are now purchased. Many complete runs are available. A well designed air-conditioned building was erected in 1952.

I come now to what is in some ways the most important feature of our work in Malaya, the Malayan Mathematical Society. Let me say at once that our work is humble but we believe of some value. The Society which was founded in 1952 is affiliated to the Mathematical Association of Great Britain. It aimed at the start to bring under its wing, teachers of mathematics at all levels whether primary or university. At deliberately low rates students in the University and in the post-school certificate classes can become members. Institutional membership is also possible. And it is pleasing to record that an active Branch has been started in the Kota Bharu Training College in the remote state of Kelantan which has now a membership of 100. The parent society itself now numbers 400.

It is plain that with such a wide range of mathematical activity to satisfy, the programme of the Society must be varied in the extreme.

Ten meetings are held each year, chiefly in Singapore. Talks are given by mathematicians at the University and the schools. Discussions take place on problems of teaching at different levels. Visiting mathematicians (very rare birds before the war) talk to us whenever occasion offers. In addition symposia are held each year (latterly with the recently formed Science Society) : noteworthy

was the convention on Mathematical education held last December in Kuala Lumpur organized by the Kota Bharu Branch and the Singapore Teachers Training College.

An exhibition of models was taken round the country by two members: the talks and exhibition proved so successful that requests for more have been received. Competitions have been started (with prizes) to encourage mathematical work at primary level and in the higher forms of schools. Slowly more teachers are being persuaded to write about their difficulties and to take an active part in the work of the Society. Many use the Problem Solving Bureau.

Finally the Society puts out—not in printed form which is beyond our means—but in mimeographed manner a monthly bulletin. It contains a wide variety of articles (some based on talks given to the Society), notes, problems, discussions and jottings. The work is heavy : much praise is due to my colleagues for their zeal : the reward is great. No teacher of mathematics in any part of Malaya, however remote and isolated (and some are indeed lonely) need feel entirely cut off from communication and discussion on topics relating to his field. For the sum of about ten shillings he receives each year some three hundred pages of mimeographed material.

The problems then which face us in Malaya are those which I described earlier, not enough appropriate textbooks, too few adequately trained teachers, a vast demand for education at all levels. Slowly some of these difficulties are being faced. Slowly curricula are being modified to meet new needs. But we need help and encouragement in our task. For this aid we look confidently towards the International Mathematical Union, certain that the wise and experienced members of this Union wish to foster in Malaya and in Singapore the great subject to which our lives have been devoted.

*University of Malaya
Singapore

INITIATION INTO GEOMETRY

By H. FREUDENTHAL

Why do we teach geometry? This is a question we have discussed for long years in the work group for mathematics of the Netherlands' section (W.V.O.) of the New Education Fellowship. Young nations have the liberty to construct an educational system of their own. The old countries are locked up in their own traditions as in a prison. Teaching programs are the main strongholds of conservative education. Modern educators who wish to avoid falling in the grooves of tradition, should re-examine the program, again and again, in details as well as on the whole.

In the years 1948-1952, we tried to draw up a new program for secondary instruction of mathematics. There has been a strong tendency towards fixation in Dutch mathematical education. When our work was finished, we had cancelled many traditional topics, and in return we had added a few new ones. But when we began, we hunted for a criterion, for general rules, in order to know whether a topic should be admitted or rejected. Mathematics has proved to be of immense use for both society and the individual. Utility is a mighty touchstone. A program that can stand all tests of usefulness looks like a mathematically proved theory.

For a long time people stuck to the formal value of mathematical education. But formal training is a dangerous argument. It is easy to justify the most old fashioned topics by formal reasons. However, stepping on through algebra, trigonometry, analytic geometry, calculus, we did not detect any aim of formal education that could not be realized within the range of a program which had been determined only by the test of usefulness.

There is one exception: Euclidean geometry. A pragmatical program of geometry could be confined to an extremely small

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group of theorems (like the Pythagorean theorem), some evident properties of similar figures, and a few formulas for circumferences, areas and volumes. Pragmatism does not call for any logical system of geometry like that granted us by the Euclidean tradition. We never thought however of abolishing geometry as an object of teaching.

Anxious to avoid vague formal arguments, we did not succeed in 1952 in unearthing the roots of our faith in geometrical instruction. Today after long years of discussion, we might better accomplish this task. I believe today we should state :

Geometry, as a logical system, is a means—and even the most powerful means—to make children feel the strength of the human spirit, that is : of their own spirit.

If this is really our goal, teaching geometry is an unparalleled strife between ideal and realization. I do not know whether a child engaged in his mathematical problems ever reached the conclusion that mathematics has been the work of outstanding human geniuses, but in any case, I am pretty sure that mathematics is rather a means to convince children of their own mental inferiority, not only in Holland, but all over the world.

In our country teaching algebra and geometry starts in the first form of the secondary educational system, that is to children aged 12 years and over. Mostly this school geometry is a watered Euclid in spite of the general conviction that children are not mature for logical rigidity at this age. This maturity must be not the pre-supposition, but the preliminary aim of the first geometrical teaching. As there are children who never reach it, there would be little sense in teasing those pupils with Euclid. There is a simple test whether the level of maturity for the geometrical system has been reached : firstly, the child must have discovered the fact of the enchantment of geometrical truth, secondly, it must have grasped the necessity of this enchantment, and thirdly, it must have been seized by this idea in such a measure that it longs seriously to proceed along the lines of the logical system.

There are many ways of initiation to geometry. The most outstanding example is the "*Ubungensammlung*" (1931) of Mrs. T. Ehrenfest-Afanassjewa, who still joins the work of our study group in spite of her great age. As a whole Mrs. Ehrenfest's system has never been practically tried out, but nevertheless it has greatly influenced Dutch geometrical education.

To quote another example: Emma Castelnuovo's well-known "*Geometria Intuitiva*" of 1948.

I will not treat these methods, which can be easily approached by all persons interested in educational problems. Likewise, I will not occupy myself with courses which still show stronger or weaker relations to classical geometrical text-books. I shall confine myself to the most striking examples of new approach to the problems of mathematical education, experimentally tested by Dutch teachers.

The first method I shall deal with is that of the text-books of Dr. W. J. Bos and P. E. Lepoeter, "*Wegwijzer in de Meetkunde*". It is the most unusual and the most remarkable course of school geometry I ever saw. I do not know any mathematical text-book where every detail has been thought out as deliberately as in the books of Bos and Lepoeter. The greatest pedagogical care has been bestowed on the step-by-step developing of the logical faculties of the pupil. The logical patterns are trained, not as abstractions, but always as structure properties of intuitive data. My exposition will become clearer, if I show you some pages from that text-book.

On the first page, which I reproduce (Fig. 1), you see a dozen problems without any text. The pupil will find all data in the figure and the question mark will tell him which angle is to be calculated. You see the pattern to be practised is that of finding out an angle by subtracting a given angle or the sum of given angles from 180° or 360° or from another given angle. The pupil shall reason in concreto; afterwards he shall put the way of reasoning in a highly schematized but nevertheless figurative symbolism, a figure as it were of the process of reasoning (Fig. 2). Words are avoided as long as possible, large use is made of the implication arrow as a notation symbol.

Here you see a few reasoning patterns, which actually will appear in more or less entangled combinations.

In order to give an impression of the way in which the method develops from the most simple to the more intricate problems, I show you a series of pages (Fig. 3-5) from the book. You see that equal lines are marked by strokes, parallels by V's, equal angles by dots or arcs, right angles by a square. (Compare, e.g. Number 39 with 48.) In order to find the angle D, the pupil must calculate angle B from the isosceles triangle ABC. Number 57 is even more involved. The pupil will try to calculate Q by means of the angles C and D; a new difficulty arises from the fact that only the sum of C and D can be known (but no more is wanted). Everybody knows that it is very difficult for a young child to develop or even to follow a composite reasoning. The ability of doing this is trained in this text-book in a thorough way. Auxiliary lines, the nightmares of geometry, are never used. Every problem can be solved by scrutinizing its data. One of the last pages of the book will give you an impression of the level the children will reach after a year. You will notice that highly complicated problems can be solved.

After having praised this method, I may raise some objections. Firstly, the lack of motivation. The child is assigned the duty of calculating an angle or proving the equality of two lines, and to put his reasonings in a fixed scheme, but never is he given the opportunity to ask why he must do so. Motivated education will encourage children to raise problems, and to formulate them, and it will stimulate the desire of solving the problems which have emerged. I fear that this method does not give enough room to this activity. Secondly, the method is too rigid. Though it is written for individual and group instruction, there is little liberty. Thirdly, most of the problems are not interesting, they have too little hold over the children, and they will not stimulate their imagination. Fourthly, I think that the systematical training of logical patterns is started too early. It may be a fault to impose an algorithm on children before they vividly feel the need of it. But in spite of these

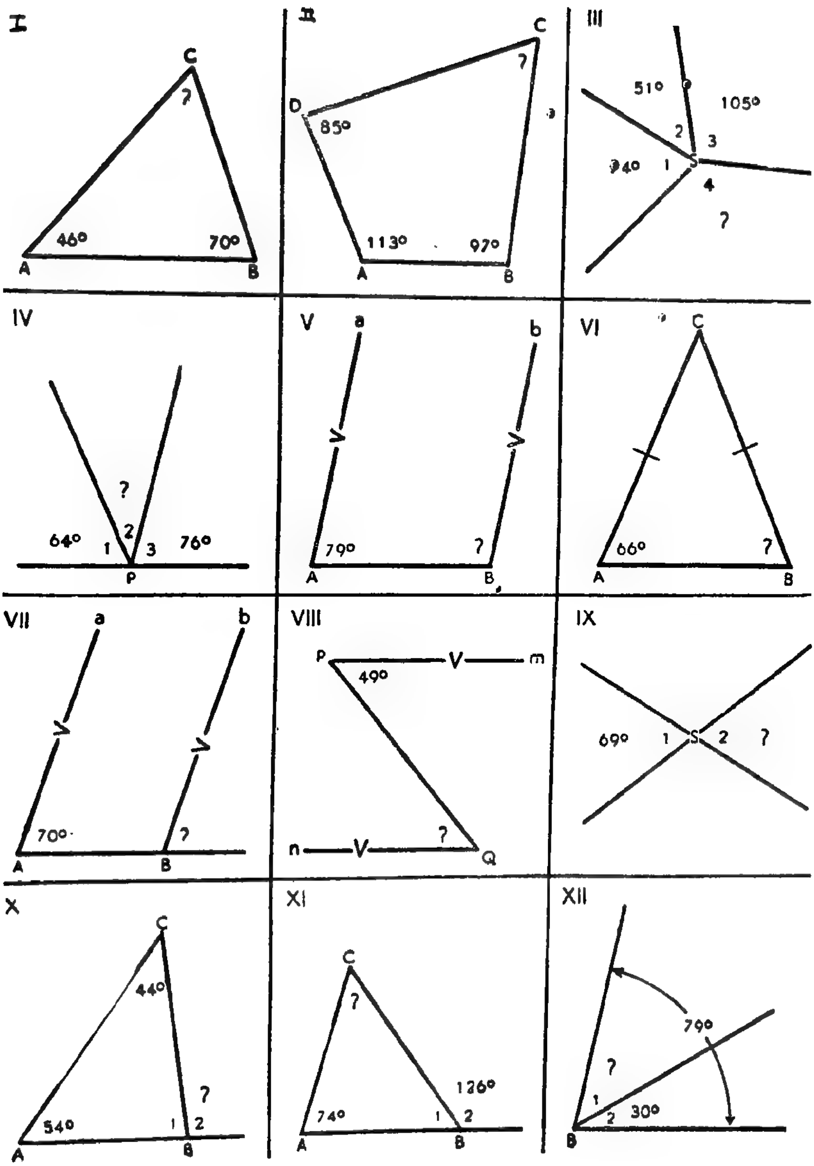


FIG. 1

To face p. 86

keten-type: $\left.\begin{array}{l} a = b \\ b = c \\ c = d \end{array}\right\} \rightarrow a = d$ (taak 21)

aanvullings-type: $\left.\begin{array}{l} a + b = 180^\circ \\ c + d = 180^\circ \\ b = d \end{array}\right\} \rightarrow a = c$ (taak 21)

aftrek-type: $\left.\begin{array}{l} a + b = c + d \\ b = d \end{array}\right\} \rightarrow a = c$ (taak 21)

optel-type: $\left.\begin{array}{l} a = b \\ c = d \end{array}\right\} \rightarrow a + c = b + d$ (taak 21)

halverings-type: $\left.\begin{array}{l} a + b = c + d \\ a = b \\ c = d \end{array}\right\} \rightarrow a = c$ (taak 25)

vervangings-type: $\left.\begin{array}{l} a = b + c \\ b = c \end{array}\right\} \rightarrow a = 2b$ (taak 25)

FIG. 2

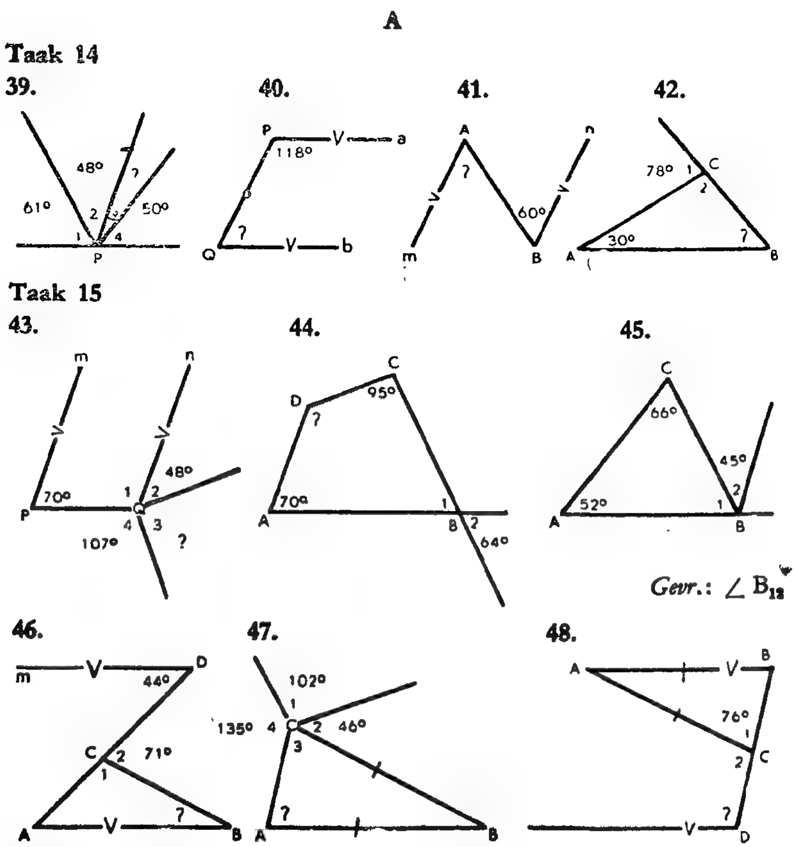
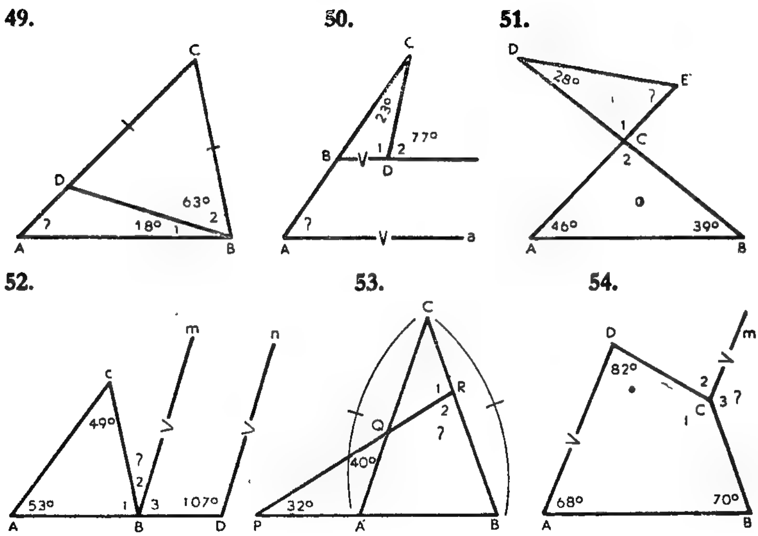


FIG. 3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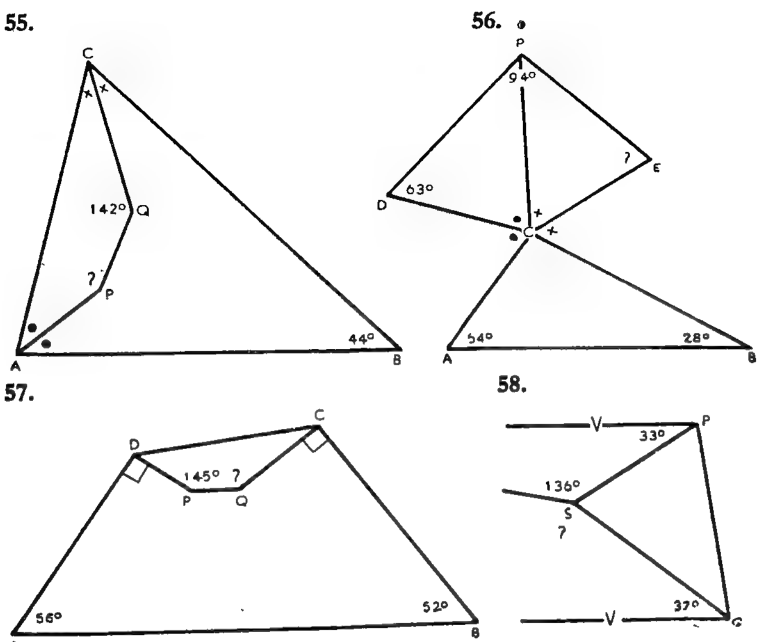
Taak 17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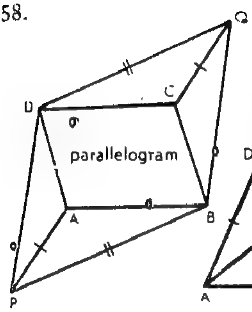
94°
?
E
63°
D
C
54°
28°
A
B
57.
C
D
145°
?
P
Q
56°
52°
A
B
58.
P
33°
136°
S
?
37°
G

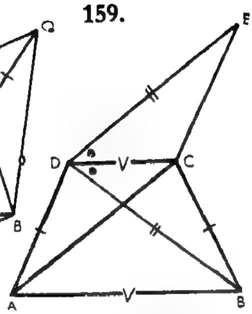
FIG. 4

158.



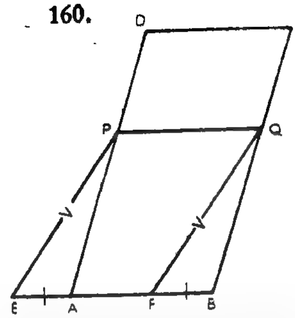
Te bew.: □ PBQD is een p.g.m.

159.



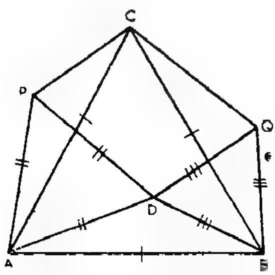
Te bew.: □ ACED is een p.g.m.

160.



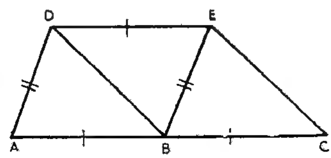
Gegeven: □ ABCD is een p.g.m.
Te bew.: □ PQCD is een p.g.m.

161.



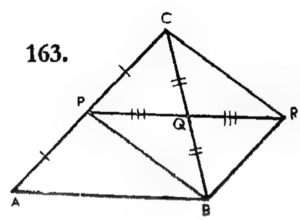
Te bew.: 1. △ APC ≌ △ ADB
2. PC = DQ
3. □ PDQC is een p.g.m.

162.



Te bew.: □BCED is een p.g.m.

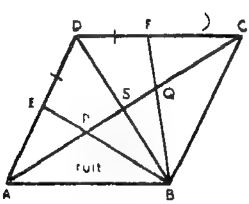
163.



Te bew.: □ ABRP is een p.g.m.

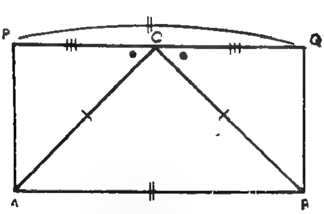
Taak 31

164.



Te bew. SP SQ

165.



Te bew.: □ ABQP is een rechthoek

FIG. 5

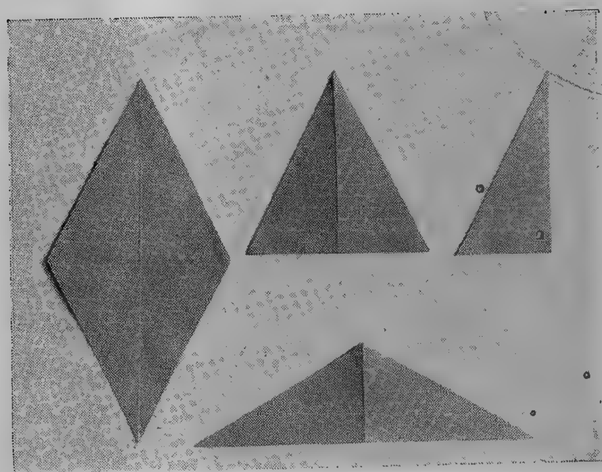


FIG. 6

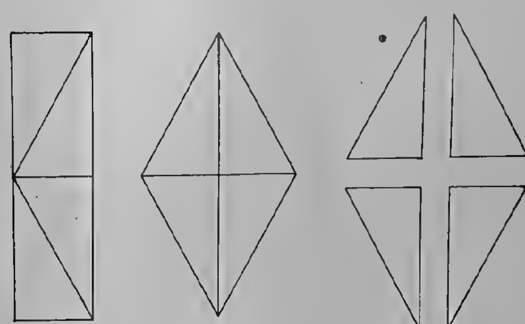


FIG. 7

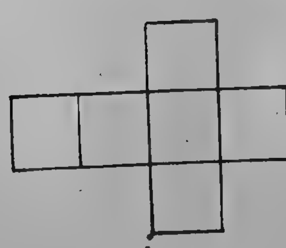


FIG. 8

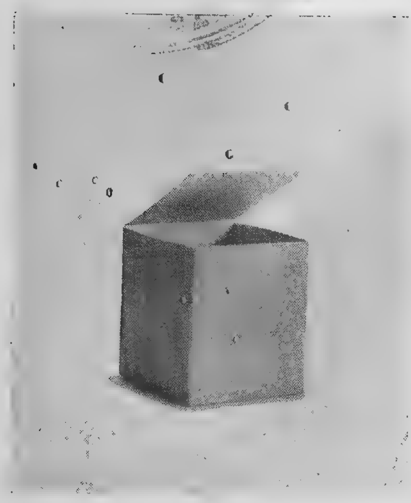


FIG. 9

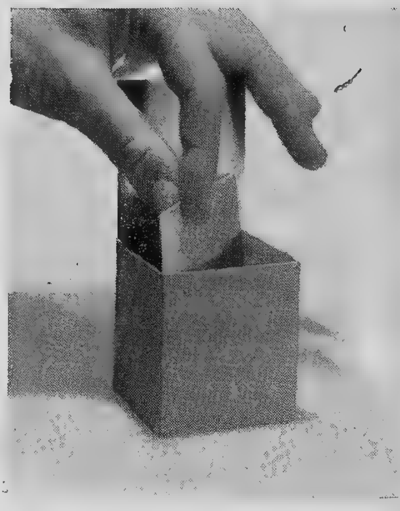


FIG. 10

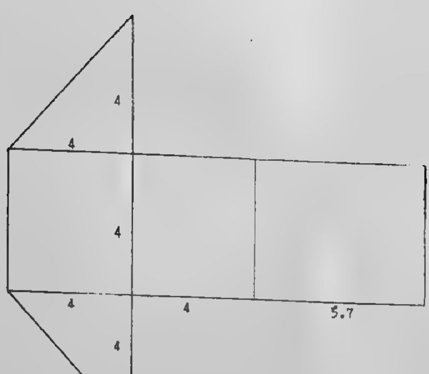


FIG. 11

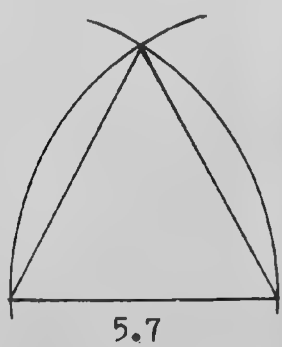


FIG. 12

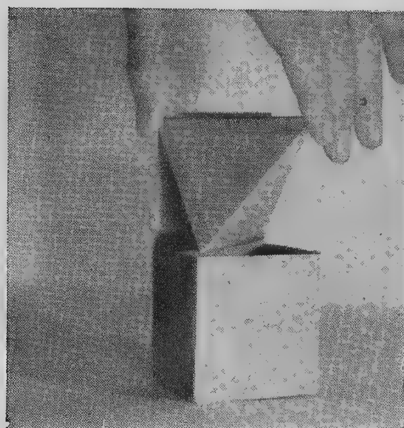


FIG. 13

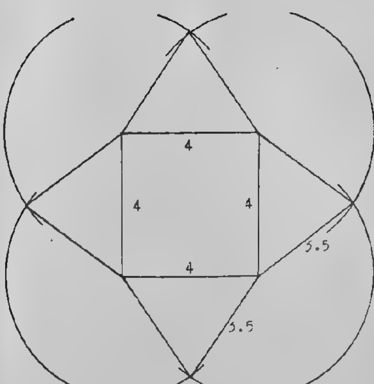


FIG. 14

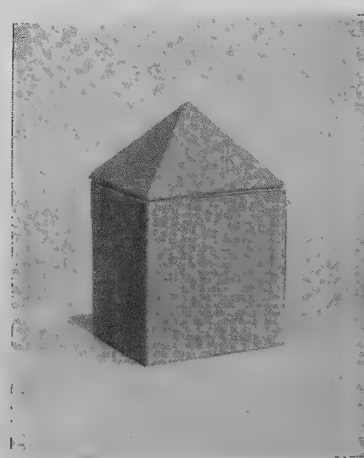


FIG. 15

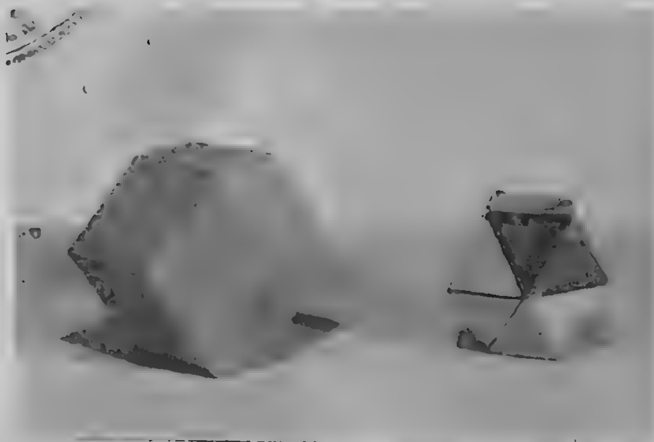


FIG. 16

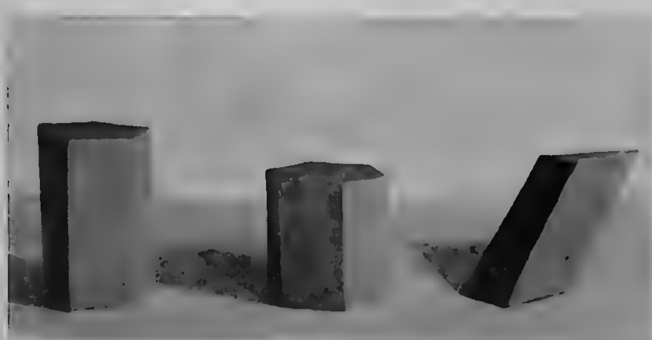


FIG 17

criticisms, I insist on pointing out the importance of this work, which is not equalled by any other I know.

The viewpoint most antagonistic to this method is supported by a group of teachers, which cannot be plainly characterized by any text-book. The procedures of this group (I may fairly speak of a school of teachers) depend heavily on the use of self-made material of a different kind, cards bearing instructions, models, and construction pieces, like Meccano parts but of a still more flexible structure.

This movement has originated from two sources. One is the work of P. M. van Hiele and his wife, Mrs. D. van Hiele-Geldof. They wrote a series of text-books (Werkboeken) especially for individual and group education, but also used in schools with a class system. In these books the traditional matter of geometrical teaching has been analyzed and arranged under the viewpoint of self-reliance of the pupil. He should not make any appeal to the teacher, until he is hampered by serious difficulties or he has accomplished a given task. During the initiation to geometry the text-book shall be superseded by the instruction card and other material, developed by the same authors, mainly by Mrs. Dr. van Hiele-Geldof. At the due moment the pupil shall switch over to the text-book.

The other protagonist of this method was Dr. P. J. van Albada, who is now a professor at the University of Indonesia at Bandung. Working independently on the same lines as the van Hieles, he inaugurated an instruction card system, which has been carried on by Dr. Miss J. A. Geldof and Mr. H. J. Jacobs Jr. Interrelations have been established between these two systems.

In Mrs. van Hiele's class-room you will notice a kind of barrow filled with portfolios, each representing a certain task and consisting of a number of instruction cards. The first card of such a portfolio will often contain an instruction, which shall puzzle the child, e.g. a problem that cannot be solved at this level, but no solution of the puzzle will be furnished until the last card of this portfolio will be reached. In any case this last card will lead to a conspicuous

piece of work that may be appreciated by the child as the crown of his labour, and which may awaken the desire of his companions to try the same portfolio.

Let us look at one of the first portfolios, called the rhombo-dodecahedron. (If you protest against terrorizing young children by verbal monstrosities, like rhombo-dodecahedron, I reassure you that in our school geometry almost all Greek and Latin expressions have been pushed away by instructive Dutch words, for the greater part invented by Simon Stevin in the beginning of the 17th century. Plain words are of a great advantage in teaching.)

Together with this portfolio the child is given a set of materials: a ruled exercise book, an unruled one, a sheet of drawing paper, a sheet of thin cardboard, a drawing triangle (set square), a pair of compasses, a ruler-staff, a pair of scissors, a knife, adhesive tape, a box containing models.

The first cards are preparatory. The child learns to erect the perpendicular at a point on a line, to make a right angle by paper-folding, to transfer a line-segment with the aid of compasses, and to draw squares, rectangles, and diagonals. He must say whether the diagonals of a rectangle are equal. He must paste a number of squares of 1 cm^2 on a rectangular card of 4 cm $\times$ 3 cm, in order to grasp the notion of area. He will lay out a rhomb with four matches, he must say whether he has seen rhombs before, and he must make rhombs by paper-folding. One question is : in how many ways can a rhomb be doubled by folding ? (Fig. 6). The child is given four congruent right-angled triangles ; he must compose them to get rectangles and to get a rhomb (Fig. 7). Which is the area of the rhomb ?

From the sixth upwards I will show you the cards in literal translation.

Sixth card : The cube.

a. Ask me for a cube (orange model). By how many squares is it bounded ?

b. How many square centimeters are needed to paste over the cube ?

Measure beforehand the edge of the cube.

c. Make the adjacent figure (Fig. 8) from drawing-paper, but as increased as to get squares with a side of 4 cm. Fold this figure so as to get a cube. (Groove the folding lines beforehand). Do not shut the upper surface (Fig. 9).

d. In the box belonging to this portfolio you will find a lot of cubes with an edge of 1 cm. They are called cubic centimeters. (cm^3). Build a cube with an edge of 3 cm from those cubes. How many do you need ? Show me what you have done. We will say : The volume of a cube with edge 3 cm is 27 cm^3.

e. Which is the volume of the cubes of numbers 6c and 6a ?

Seventh card : The prism.

a. Make a rectangle of 5.7 cm × 4 cm. It fits into the cube you have made in number 6c, in such a way that the cube is divided in two equal parts by this partition wall (Fig. 10). Every part will be called a prism.

b. Draw the figure on drawing-paper (Fig. 11). It consists of two squares with a side of 4 cm, a rectangle of 4 cm × 5.7 cm and two triangles. This figure is called the reticulation of the prism of 7a, because it can be folded so as to get this prism. Cut it out, groove the folding lines, and make the prism.

c. How many cm^3 is the volume of this prism ?

Eighth Card : The tetrahedron.

a. A cube is also called a hexahedron. Why ? There is also a tetrahedron. It is bounded by four triangles. Draw a triangle with ruler and compasses like that in the adjacent figure (Fig. 12).

b. Draw a figure that may be folded in such a way that you get a tetrahedron with an edge of 5.7 cm. (If you do not succeed, look at the orange model number...). Show me the figure before cutting it out.

c. Cut out the reticulation, groove the folding lines, and make a tetrahedron of it.

d. This tetrahedron can be put in the cube of 6c, so that the lid can be shut. Try it.

[Note for the reader : This is just one of the two tetrahedra formed by six surface diagonals of the cube (Fig. 13).]

Ninth Card : The pyramid. (Orange models 5, 6, 7.)

The adjacent figure is the reticulation of a pyramid (Fig. 14). The side of the square is 4 cm, and from the corners 3.5 cm have been circled round.

a. Draw this reticulation and make the pyramid.

(The following tasks will be made by three children together.)

b. How many of these pyramids are wanted for building a cube ? Make as many as you need, and build a cube as big as that of No. 6c.

c. How many cm^3 is the volume of the pyramid ?

d. Stick fast the cube, and paste one pyramid upon each surface of the cube. It becomes a rhombo-dodecahedron. Can you explain the name ? (Fig. 15-16).

e. How many cm^3 is the volume ?

f. Ask me for the rhombo-dodecahedron (Orange model No....), and make one from thin cardboard.

Tenth Card :

Learn what follows :

a. A quadrangle with four equal sides is called a rhomb.

b. A rhomb can be doubled by folding along its diagonals.

c. The diagonals of a rhomb form right angles.

Answer the questions :

d. In how many ways can a square be doubled by folding ?

e. Can you double a rectangle by folding it along a diagonal ?

f. Can you double a rectangle by folding it in another way? (Make a square and a rectangle.) Tell me!

g. Ask me for the orange models Nos. 2, 4, 5, 6, 7, 8, 10, 1. (Figs. 17-18). Tell me the names.

Read over this portfolio and your exercise book, and ask me for a test card.

Let us now analyze the most important features of this chapter.

1. All stress is laid on stereometry, not by pure chance, but intentionally. One may argue that geometry teaching should start with stereometry. Except one example—I shall deal with later on,—there are no striking relations of figures in the plane to start geometry with. All properties of figures in the plane are either too silly (e.g. congruence or parallelism—which cannot cause any questions at this stage) or too sophisticated (e.g. Pythagorean theorem, similarity).

2. Solid bodies are less abstract than plane shapes. The child can grasp them in a literal sense. Stereometry meets the children's creative wishes. Figures are drawn, solids are made.

3. The children become acquainted with space. You know how many children, who were keen in planimetry, get in serious difficulties as soon as stereometry starts. Their intuition and imagination have been irreparably wasted by three or four years' exclusive planimetric teaching. It has been established by experience that nobody who has started geometry with space, will meet any serious obstacles, when passing to stereometry in the systematic course.

4. The notions of area and volume have been introduced neither by abstract reasoning nor by bluntly appointing that the area is to be length-times-width, and the volume length-times-width-times-height, but by fitting a number of things into one thing—the only psychologically justified manner to introduce them.

5. Fitting is the leading psychological idea of this portfolio, not only for the definition of area and volume. The edges of a

reticulation *match* one another. Two prisms *fill* a cube. The partition wall *fits* into the cube, and the tetrahedron of the six surface diagonals *does likewise*. The six pyramids *fill* the cube, and placed upon its surfaces their sides will pair-wise *merge* into each other—you can feel it by stroking along the surfaces.

Fitting is a motor sensation. Psychologists can tell you how strongly the motor component of the personality is marked at this age, how important motor apprehension and memory may be.

6. Things fit. Do children ask why? Apart from a rare exception they do not. All these miracles of our space do not seem to make any impression. But they grind as mill-stones. The highest pedagogical virtue is patience. One day the child will ask why, and there is no use to start systematic geometry before that day has come. Even more : it can really do harm. For we have agreed upon teaching geometry as a means to make children feel the strength of the human spirit—that is of their own spirit—and we should not deprive them of the right to make discoveries of their own. The clue of geometry is the word "why". Only joy-killers will deliver the clue previously.

7. The miracles of fitting are to prepare maturity for systematic geometry. But if it has been reached they will not become redundant. They will furnish raw material for geometrical thinking, and they will be tokens of the past, when a higher level will have been reached. This is a necessity in mathematical education. The child shall re-call and re-consider the treasure of old problems and re-examine the old solutions at each new stage. In this way, he will be able to judge his own development and his own progress, by retrospective views, as somebody who turns over the leaves of a photograph album, or as a mountain climber who casts a glance back on the way he has gone.

8. It hardly needs stressing that a portfolio like the one analyzed, though a system of instruction, grants a high measure of liberty to the child that works with it. Modern education endeavours to develop not only rational thinking, but also imagination. Mathematics

cannot do without imagination, nor can mathematical teaching. I hope you have felt the strong appeal to imagination in this method.

Let us come back to more special questions. I announced that there is one way to start intuitive geometric teaching with planimetry. The method I have in view is : paving a floor with tiles. This is an idea of van Albada worked out by van Hieles. The slides I shall show you reproduce partially an elaboration by Dr. de Miranda.

Somewhere in the school building a tiled floor or wall will be found. The children get the task of copying it. They may vary the pattern (Fig. 19). There are many ways to pave the plane with square tiles. Somebody will try it with triangles (Fig. 20). The others join him. The triangles are varied. Any kind of congruent triangles will do it. Any kind of congruent quadrangles too ? Yes ! (Fig. 21). Pioneers will explore the pentagon. But the pentagon is obstinate (Fig. 22). It does not work. Four is the upper bound. Is it ? Yes, it is. For five cannot succeed. Are you sure ? If five is bad, six is worse. An interruption : It works though. The lavatory is paved with regular hexagons. Many kinds of hexagons can be used for paving the plane (Fig. 23).

Why do some polygons fit, and why don't others ? By round about ways the solution will be approached. If the reason is instinctively felt, it will still be difficult to formulate it. Triangles fit because the sum of their angles is half a turn—180° : quadrangles because the sum is a full turn—360°. Pentagons will have one and a half turn—this is an awkward matter. Hexagons give twice a turn, if I can select two angles that sum up to a full turn, I get a chance.

The leading idea of this series of lessons is once more the property of fitting, now applied to the plane. There is but one problem of fitting in the plane, and that one is paving. This problem has been treated here in an extensive measure.

The theorems about the sum of the angles of a polygon have appeared here in a natural way. Not as theorems, but as important

problems. Not as abstract truth, but as incisive relations, which make fitting possible or impossible. Summing up the angles of a polygon has emerged, not as an arbitrary invention of man, but as a necessity of nature. Even congruence has lost its character of dead abstraction.

There is still another reason to prefer tile-paving to the rhombo-dodecahedron. In the portfolio " rhombo-dodecahedron " all things fit. Here there is one bad fellow : the pentagon *that refused to act.* With the result, that even the laziest is obliged to ask : why ? The children are not guided along a smooth road of thinking, but plunged into dialectics.

After having laid out a floor of equilateral triangles the children make drawings (Fig. 24). A minority proceeds without a fixed plan ; the triangles are attached one to another at random. The majority build horizontal strips of triangles. It will be a very rare exception if a child discovers the three systems of parallel lines and performs the construction in an economic way. There is little doubt that the first two groups will not be mature at this moment for systematical geometry.

Let us cast a glance into some other portfolios. One is called " The right-angled triangle ". It starts with two puzzles. A set square moves between two pins fastened into the paper (Fig. 25). Which is the locus described by the vertex of the right angle of the set square ? The child will discover it is half a circle or a whole circle. He may ask why or he may not. In any case no proof shall be given. One proceeds to the second puzzle, the famous problem of Menon's: doubling a square. He will not succeed. Now he is given a great number of isosceles right-angled triangles (with sides 2 cm). Can you make a square from five triangles ? With how many can you ? Write down the sequence of these numbers (Fig. 26). Can you guess how it would continue if I should give more triangles ? How big are their areas ? Of which squares can you tell me the length of the sides ? Try again to solve the second puzzle !

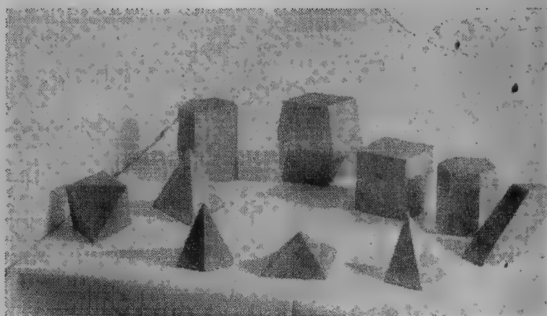


FIG. 18

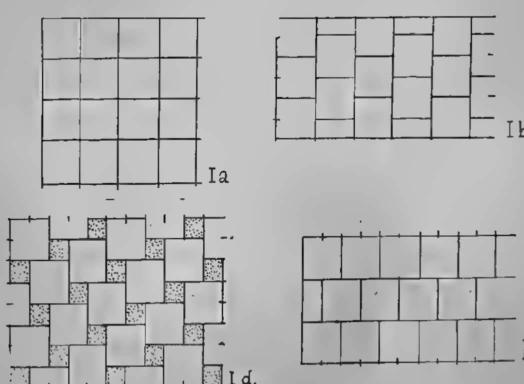


FIG. 19

To face p. 94

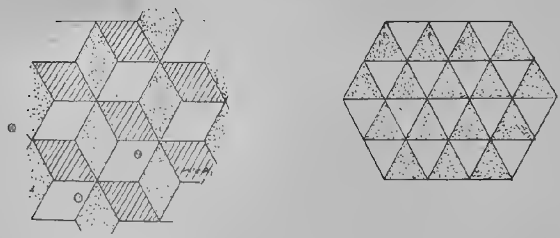


FIG. 20

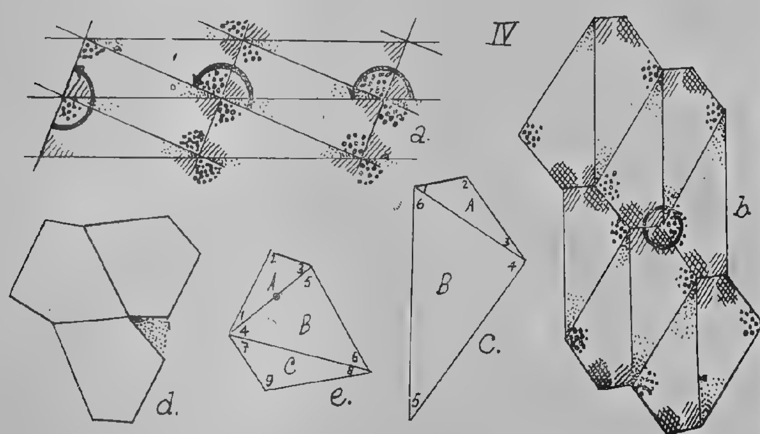


FIG. 21

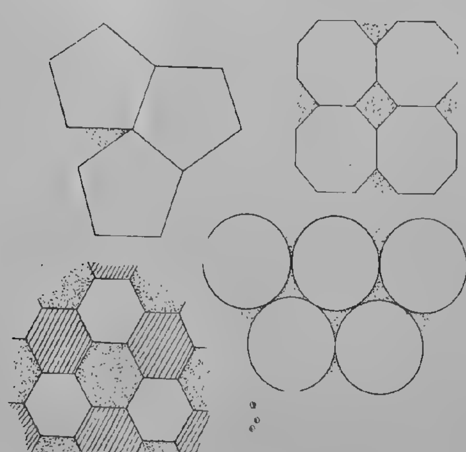


FIG. 22

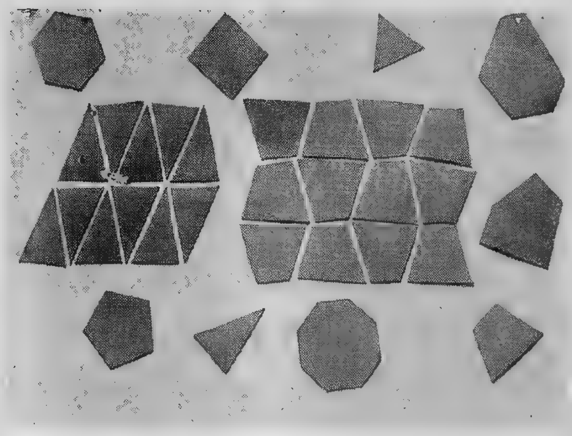


FIG. 23

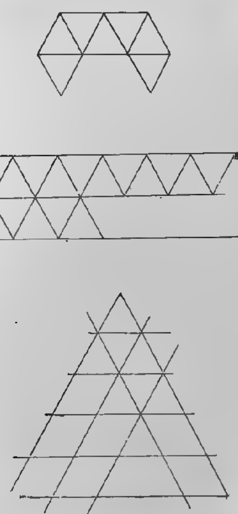


FIG. 24

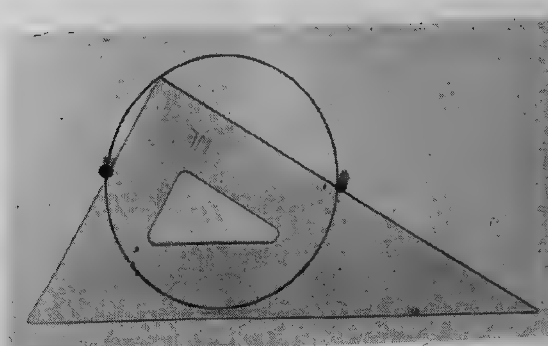


FIG. 25

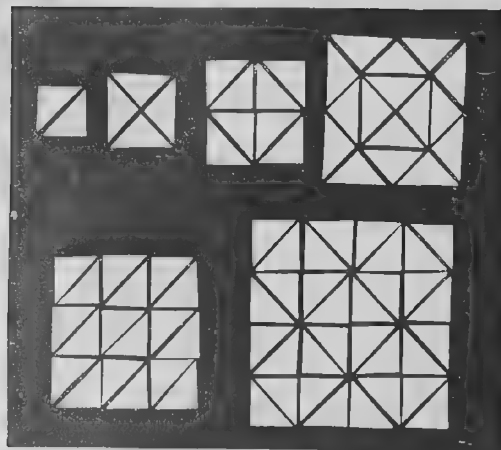


FIG. 26

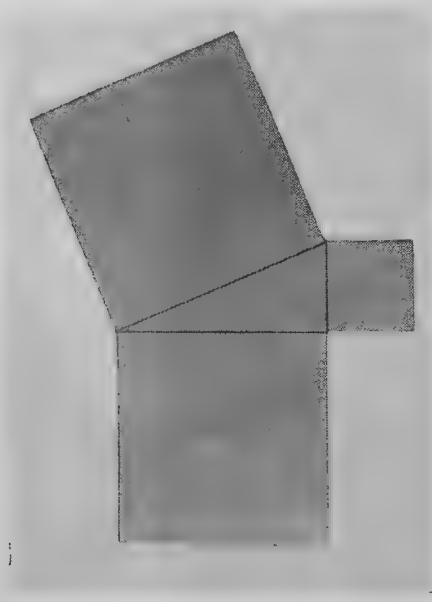


FIG. 27

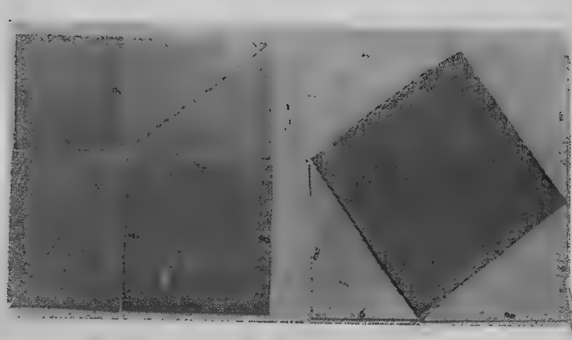


FIG. 28


WALCHEREN

FIG 29

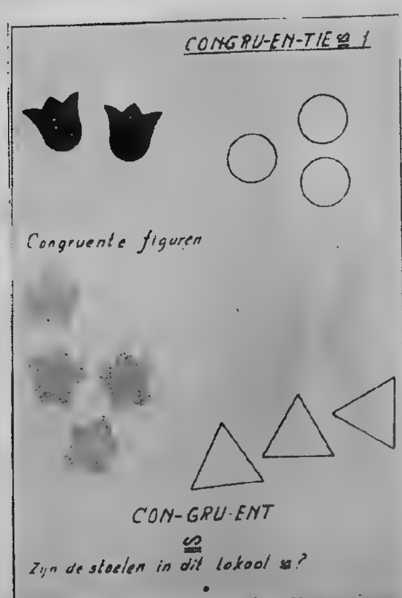

CONGRU-EN-TIE ≌ 1
Congruente figuren
CON-GRU-ENT
≌
Zijn de stoelen in dit lokaal ≌?

FIG. 30

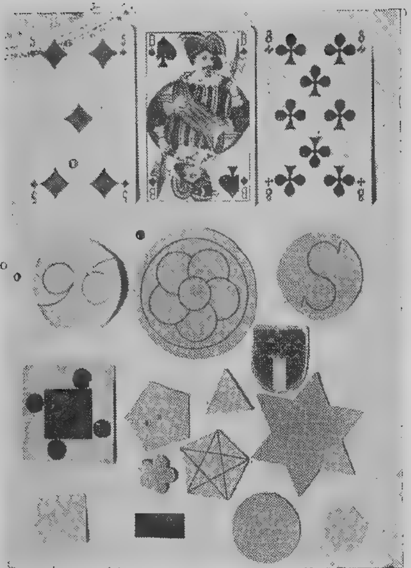
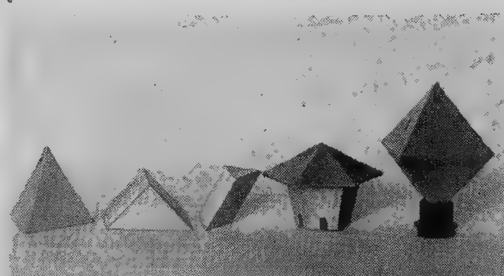


FIG. 36

FIG. 37



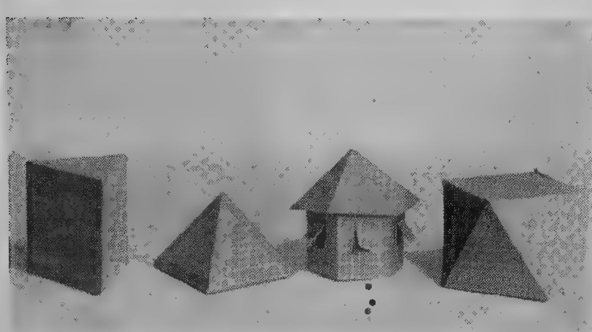


FIG. 38

After this the child is given a figure that represents Pythagoras' theorem (Fig. 27). He is asked to hunt for a proof. He will not succeed. He is given two boxes, which contain respectively the pieces needed to compose the two Figures 28a and b. Gradually the child will be guided to find the so-called Indian proof of the Pythagorean theorem. As a consequence a lot of theorems about triangles will be developed and the labour will be crowned by the proof of Thales' theorem, the puzzle he started with.

Another portfolio is called: Enlarging and reducing. Similarity is not introduced by a clear cut definition, but by two examples (Fig. 29).

Figures such as the two facades are called similar. Look at the two maps of the province Zeeland. The scale of one is 1 : 400,000. How long is the distance from Flushing to Bergen op Zoom? No scale is mentioned on the second map. Can you determine it? Let me see it. Determine similarity factor of the two facades.

In the sequel similarity and congruence will be treated in a more systematic manner. Similarity precedes congruence, not by chance, but on good grounds: congruence can be established at sight, whereas similarity puts problems: to determine the similarity factor and to find out criteria of similarity. Likewise similarity of quadrangles and other figures is treated before that of triangles, because the latter case is too special. Indeed, similarity of triangles can be decided by bare arguments about angles, one can dispense with proportionality of sides, and this is a serious disadvantage.

An interesting portfolio treating congruence and similarity is contained in the system of Dr. J. A. Geldof. It proceeds more slowly than that of the van Hieles', but the material defining the fundamental notions is more extensive. I cannot treat this portfolio at full length, but I shall show you the first card (Fig. 30) introducing congruence, other material (Fig. 31) illustrating it, and one card introducing similarity (Fig. 32). This portfolio finishes by a review of solids (Fig. 33) where the child is asked to pick out congruent

and similar samples, and by a few questions, partially related to the kites of Fig. 34 :

a. The kites D and E are similar. The kites D and F are not similar. Tell me why not? Take the kites with you.

b. Draw a kite similar to F.

c. Draw two similar quadrangles.

d. Can you draw two non-similar quadrangles ? (Orally).

e. Are circles always similar ?

f. Are rectangles always similar ?

g. Mention figures that are always similar.

I am sorry I cannot show you van Albada's astonishing material about perspective, which leads the twelve-year olds in a natural way up to fairly complicated designs of descriptive geometry. Van Albada has made the important discovery that the proper place of descriptive geometry in geometrical instruction is not the higher but just the lower level.

As a last example I will draw your attention to one of the most exciting topics which is met in all these systems of initiation into geometry—the phenomenon of symmetry. Ornaments from the dawn of mankind seem to support the thesis that symmetry has been the first geometrical idea that arose in the human mind, and which was given expression in human handicraft. Through an infinity of instances from nature and technics, symmetry may be approached by the childish spirit. It is a remarkable fact that planar symmetry with respect to one axis or a number of axes is a strikingly evident notion whereas many people never really become acquainted with central symmetry, even though they learn mathematics. Few children discover of their own the congruence of the triangles into which a general parallelogram is divided by a diagonal. The only exception are those who have made up parallelograms from triangular tiles and those who have been trained in central symmetry.

Without further commentary I shall show you some cards and models used in the treatment of symmetry (Fig. 35-38). It is worthwhile mentioning the use made of mirrors as a tool in this chapter.

When drawing my final conclusions I must compare once more the two systems I considered as antagonistic approaches to initial teaching of geometry. The system of Bos and Lepoeter is much more worked out in all details than the other ones. It has been laid down in a textbook accessible to as many teachers as you want, whereas the other methods are confined to comparatively small groups. The first system can be used with real success by any teacher who will make fair attempt to give good instruction. The latter system supposes a depth of psychological background that is not provided by our profoundly scientific, but pedagogically shallow teacher training. In every special case the choice of system (which may be extreme or intermediate) will heavily depend on the teacher's personality structure.

Utrecht, Netherlands

ON MATHEMATICAL EDUCATION IN THE U.S.S.R.

By A. D. ALEXANDROV

In this brief report a general scheme of mathematical education in the U.S.S.R. will be outlined with special stress on university education.

1. In order to comprehend some specific features of our system of education let us cast a glance at its history.

Though before the Revolution, mathematics in Russia was at a high international level, the extension of education was too miserable. Illiteracy was a great evil of that time. The majority of people had no access to education. Gymnasiums and universities were accessible only to the elite. Just immediately after the Revolution our task was to change radically this situation. We did our best and development of primary, secondary and higher schools was immensely remarkable, the number of students was raised to an extremely high level. Much attention was paid, for instance, to the development of education in the republics of Middle Asia. It was necessary to ensure instruction in native languages there.

These conditions demanded, most urgently, a uniform and strict system in order to implement a higher standard for the growing number of secondary and higher schools everywhere. Life develops in a quick way and there arise new problems for which a better solution needs to be found. Certain efforts are being undertaken aimed at fulfilling these tasks.

The general system of education in the U.S.S.R. can be presented as follows.

2. Middle school, which takes ten years. Children enter the school at the age of 7. To-day we have compulsory course for only seven years. After seven years of learning a boy or a girl can

This lecture was given at the South Asian Conference on Mathematical Education held on 22-28 February 1956 at the Tata Institute of Fundamental Research, Bombay.

according to their desire continue their studies at school or enter a technical middle school. There are different types of such schools: engineering, technology, agriculture and so on; they have as a rule 3 or 4 year courses.

A ten year course of education is being put into force mainly in big cities to-day. In five years, however, compulsory ten years' education will be introduced almost everywhere. Native language is the means of instruction and each nation however small it might be has its own schools. Teaching in middle school is based upon a unified programme. As far as mathematics is concerned the programme contains arithmetic during the first five years of learning, then algebra and geometry, and trigonometry at the eighth year. Algebra is taught up to complex numbers, elementary functions, and the theory of limits, geometry up to polyhedra and solids of revolution.

Children who take a keen interest in mathematics have an opportunity to attend special circles at schools and universities. In order to attract more attention to mathematics and encourage gifted children many universities arrange annual school mathematical competitions, so called "mathematical olympics". A school boy should solve in such a competition, a number of complicated problems which, however, do not surpass the limits of the school programme. Those who have managed successfully are stated to be winners of the competition and are also encouraged with prizes.

Mathematical olympics have gained a broad acknowledgment among boys and girls. For instance annually 1000 take part in the olympics of Moscow university.

3. Institutions of higher education. In order to enter any higher school one has to pass entrance examinations. And though the Soviet Union has about 800 higher schools with more than $1\frac{1}{2}$ millions of students, we have even more those who want to obtain higher education. It allows, especially in the biggest higher schools, to choose through entering examinations a good composition of students. Each student who is successful in his study

has a scholarship and therefore no criterion but personal abilities determine the possibility of obtaining higher school education for anybody. There is also a highly developed system of correspondence and evening courses designed for working people who want to study. The course at the Soviet higher schools lasts five years as a rule, four years at pedagogical institutes. After each term a student has to pass examinations in 3, 4, or maybe in two subjects. Higher mathematics is a special subject of study at universities and pedagogical institutes which train middle school teachers.

Mathematical training at higher technical colleges has aims and problems of its own. One of them is to adapt it to the needs and programmes of studies of engineering, technology, physics, and other special subjects.

A student at technical college deals with mathematics during the first two years. The general course of mathematics consists of analytic geometry with vector algebra, elements of higher algebra, calculus, differential equations, power and Fourier series. Short additional courses on the theory of complex functions, probability theory, partial differential equations, are given after the general course. Physics departments of universities have their own more extensive programmes of mathematics.

4. UNIVERSITIES. There are 33 universities in our country. Almost all of them have their physical mathematical faculties, while the bigger ones like Leningrad University, for example, have a separate mathematical faculty including three departments, that of pure mathematics, of mechanics and astronomy. This separation is justified, at least, by the fact that the mathematical faculty of Leningrad University accounts for more than 1000 students.

The general curriculum and programme of university education are common for all universities of the U.S.S.R. except the bigger ones. Such universities with a large and highly qualified teaching staff outline their own curriculum within the general content of university education.

The task of university education is aimed at training teachers of middle and higher schools and scientific workers. A graduate does not obtain any special degree but his qualification as a teacher and scientific worker is defined in his diploma.

5. It is quite obvious that a scientific worker would not come out of *each* student. But university education should initiate each student to be engaged in scientific work. A university should provide all the opportunities for each capable student to fulfil a research work after graduation.

The problem of combination of a necessary middle level with the development of individual abilities has been already discussed at this Conference.

Its solution is assured by two methods. The first one is a due combination of teaching with research work. The university must be a school and research institute at the same time. Only a lecturer experienced in science can give to his students not only a dry mummy but a sound and living science. Only such a teacher can make the students think, not only learn. Therefore we consider the combination of teaching at the universities with research work to be absolutely indispensable and compulsory. Much effort is applied to realize this principle.

The second method provides a due structure of curriculum and various methods of involving the students in scientific work.

In this connection we have two new problems :

a. Reasonable combination of abstract theories of pure mathematics with applications to mechanics and physics.

b. Reasonable combination of a sufficiently wide basic mathematical education with a certain specialization.

A student naturally should have a good experience in all branches of mathematics. But he will fail to deeply penetrate into all those branches, while the approach to research work cannot be gained without such penetration. This fact causes the problem of

combination of a wide basic preparation with specialization which brings a student to the border where some research work begins.

Only combination of abstract generalizations with concrete knowledge resulting from natural science and technique will allow a student to find a true orientation in mathematics. The beautiful tree of mathematics grows up on the soil of natural science. Growing to the top of abstraction, it bears fruit, the seeds of which drop on the same soil of natural science and technique. It means not only connection with applications. We are also speaking of pure mathematics of higher style.

Great mathematicians of all times have been developing their fruitful ideas in an indissoluble connection with problems of the exact natural science. One can draw these facts from works of Archimedes, Newton, Euler, Lagrange, Gauss, Riemann, Poincare, Hilbert. The progress of mathematics is closely connected with the progress of exact natural science and technique. Mathematical education and research has to pay much attention to the problem of physics and technology.

6. What is the curriculum at the department of pure mathematics of Leningrad University, approved by the learned council of the faculty ?

(i) Analysis in the 1st and 2nd years.

(ii) Differential equations, integral equations and the calculus of variations, with elements of functional analysis in the 2nd, 3rd and 4th years.

(iii) Courses in the theories of real and complex functions in the 3rd year.

(iv) Algebra with linear transformations and elements of group theory, and number theory in the 1st and 2nd years.

(v) Geometry, analytic in the 1st year, differential in the 2nd, and the foundations of geometry, in the 3rd year.

(vi) Theory of probability in the 3rd year.

(vii) Mechanics in the 1st, 2nd, 3rd years,

THE CURRICULUM OF THE DEPARTMENT OF (PURE) MATHEMATICS OF THE MATHEMATICAL FACULTY OF LENINGRAD UNIVERSITY

Subject	Time: No. of hours			Distribution of hours a week									
				I Year		II Year		III Year		IV Year		V Year	
	Lectures	Exercises	Laboratory	1st term	2nd term	1st term	2nd term	1st term	2nd term	1st term	2nd term	1st term	2nd term
1. Analysis	255	220	—	8	8	6	6	—	—	—	—	—	—
2. Analytic geometry ...	100	90	—	7	4	—	—	—	—	—	—	—	—
3. Higher algebra	108	100	—	4	4	4	—	—	—	—	—	—	—
4. Physics	200	36	52	—	—	—	—	6	7	2	2	—	—
5. Theoretical physics ...	76	—	—	—	—	—	—	—	—	—	2	4	—
6. Astronomy	68	—	—	—	—	2	2	—	—	—	—	—	—
7. Differential geometry ...	68	32	—	—	—	—	3	3	—	—	—	—	—
8. Foundations of geometry	50	—	—	—	—	—	—	—	—	—	3	—	—
9. Differential equations ...	70	70	—	—	—	4	4	—	—	—	—	—	—
10. Mathematical physics ...	100	34	—	—	—	—	—	2	4	2	—	—	—

11.	Mechanics (including continuous media)	220	135	—	—	6	6	5	4	—	—	—	—	—
12.	Real functions	68	—	—	—	—	—	—	2	2	—	—	—	—
13.	Complex functions ...	70	30	—	—	—	—	—	2	4	—	—	—	—
14.	Theory of probability ...	50	—	—	—	—	—	—	3	—	—	—	—	—
15.	Integral equations ...	34	—	—	—	—	—	—	—	—	2	—	—	—
16.	Calculus of variations ...	50	—	—	—	—	—	—	3	—	—	—	—	—
17.	Number theory	32	—	—	—	—	—	2	—	—	—	—	—	—
18.	Computing machines ...	50	—	14	—	—	—	—	—	4	—	—	—	—
19.	History of mathematics	70	—	—	—	—	—	—	—	—	2	2	—	—
20.	Mathematical practice (problems, computations etc.)	40	150	50	2	—	—	—	3	3	3	—	—	—
21.	Methods of teaching and practice at school ...	100	30	—	—	—	—	—	—	—	6	2	—	—
22.	Special courses and seminars to be chosen ...	220	100	—	—	—	—	—	—	2	6	6	7	—

Philosophy and foreign language omitted

(viii) Physics, a general course as well as a special course of modern theoretical physics, in the 3rd, 4th, 5th years. Physics is taught at this period in order to have no difficulties with mathematics and not to repeat the foundations of mechanics.

(ix) Astronomy in the 1st and 2nd years, and then such courses as those on computing machines, or the history of mathematics, or methods of education and philosophy.

In the middle of the 3rd year there are introduced, step by step, some special courses which a student is allowed to choose according to his desire.

Thus a student who wishes to study algebra has an opportunity to attend lectures on the theory of groups, Galois theory, continuous groups or some other subjects. A student who joins the chair of geometry may study topology, Riemannian geometry, and so on. Besides, professors give lectures on their own subjects of research, and thus a student has an opportunity to be acquainted with up-to-date results. Students may also join special seminars, for instance in mathematical logic, and to work there with post-graduate students, lecturers and professors.

The very existence of such seminars as well as the programmes of the special courses are determined by the scientific activities and interests of professors only. As far as their courses and seminars are concerned, the professors are entirely free in the choice of subjects and programmes, their only task being to promote sufficiently the interests and activities of students.

In order to fulfil the same task students have to prepare small papers at the 3rd and 4th years. These papers may contain a short survey or a solution of certain problems.

7. In the 1st and 2nd years the students may join the circles where they study some chosen additional subjects or get an additional training in solving problems above an average level.

All kinds of scientific activities of students are supported by the Students Scientific Society. The Society arranges students'

conferences, competitions in solving problems, issues special wall newspapers. We have prizes for the best solution of problems and for the best scientific paper. Each student who is successful in his study gets a scholarship. The best students get scholarships of honour.

In the last course of learning a student is almost free from obligatory lectures the main task of his being to attend special courses, seminars, and above all, to prepare the diploma work. This work has to be a small scientific paper. The level of its originality, of course, depends upon the ability of the student.

As a result of the whole system we have every year a number of students who have papers worthy to be published in mathematical magazines.

8. For the sake of the further training of scientific workers and higher school teachers and lecturers a wide-spread system of post-graduate studies has been set up at the higher schools and research establishments. In order to enter a post-graduate course, a graduate has to pass examinations. The graduates recommended by the learned council of the faculty are allowed to enter the post-graduate course immediately after their graduation, while all the others after at least three years of work in their speciality.

Each post-graduate has a consulting professor who accepts his post-graduate according to his own opinion.

The post-graduate course lasts three years. A post-graduate is obliged to pass two examinations in his speciality, (the subjects of the examinations being prescribed by the consulting professor), a foreign language and philosophy. But the main task of his is to be engaged in research work which might be accepted as his Candidate's thesis. The Candidate's degree is, or seems to be, above the Master's degree in England.

This post-graduate system is the main source of the higher school teachers at the level of a lecturer. Due to all the universities we

have now the necessary number of lecturers with the scientific Candidate's degree.

9. It seems that the course on the history of mathematics given at our universities can play an important role in broadening the students' understanding of mathematics. This course gives a student a connected survey of basic mathematical concepts and theories in their birth and development, in their historical sequence and logical connection. It leads to a better understanding of the sources and foundations of mathematics. In particular, the history teaches us that modern mathematics as developed during the last centuries by European scientists has its source not only in Greek geometry as is often said, but in no less a degree in the concept of number which had sprung up in India.

What the Greek genius could not attain, i.e. to give the most convenient notations of numbers, to extend the system of numbers, and to abstract irrational numbers from their geometric basis—all this was done at the first stage in India and reached Europe through Middle Asia and Arabic countries. For instance one can find in the writings of Eastern mathematicians, four centuries before Newton, the same definition of number which was given by Newton, in his Arithmetic.

India was a cradle of one of the most fundamental concepts of mathematics. The glorious history is a guarantee of the bright future of science.

Leningrad University

NEW MATERIAL AND A NEW METHOD FOR THE TEACHING OF ELEMENTARY CALCULATIONS IN PRIMARY SCHOOLS

By GUSTAVE CHOQUET

[In this lecture Professer Choquet described Mr. G. Cuisenaire's method of teaching arithmetic in primary schools. Ref. G. Cuisenaire and C. Gattegno: *Numbers in colour*, William Heinemann, 1954. Cf. footnote on page 41 of this *Report*.]

REPORT ON MATHEMATICAL INSTRUCTION IN ITALY

By ENRICO BOMPIANI

Before entering on the subject proper of this lecture, namely mathematical instruction in Italy, I think it advisable, or better necessary, to give you a general picture of the educational system in Italy so that, when introduced to details of mathematical instruction, you may understand the different aims of the various kinds of schools, and of their various levels, and see how mathematical teaching is adapted to them.

Let me point out some difficulties which make themselves immediately apparent when one tries to sketch the educational system in a definite country at a definite moment.

Educational systems differ widely in different countries. They are the final product (at a certain date) of different cultural traditions, of political, geographical and economic conditions : moreover, where culture is really alive, educational systems are continuously changing in order to profit by the most recent scientific acquisitions and to adapt themselves to the varying conditions of the country.

This makes it evident that no system can be really understood without knowing, at least in a general way, its historical background, and that the description of the system at a definite moment completely disregards the fermentation of ideas which will produce the next system. In addition, because of the variety of educational systems in different countries, it is even impossible to translate in a different language the technical dénominations adopted in a country. However, since it is necessary to give a translation from Italian into English this can only be an approximate one, and it

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is only meant to convey some ideas referring to similar institutions : that is why Italian denominations are also given for exact reference.

This report is divided in two parts. The first gives a general sketch of the present educational system in Italy (different types of schools and their aims ; the Ministry of Public Instruction and its role ; compulsory education and fees ; historical survey of the preceding educational systems). The second part is only devoted to mathematical instruction : programmes, aims, and related information is given for the different types and levels of schools.

Part I

The Present Educational System In Italy

1. Types of schools at different levels.
2. The Ministry of Public Instruction. Its role.
3. Compulsory education. Fees.
4. Historical survey of the preceding educational system.

1. Types of schools at different levels.

The educational system in Italy is organized in the following categories or levels :

I. SCUOLA ELEMENTARE O PRIMARIA : ELEMENTARY OR PRIMARY SCHOOL.

Children enter this school at the age of 6 ; it lasts *five years* (from 6 to 11 years of age).

II. SCUOLA SECONDARIA INFERIORE : LOWER SECONDARY SCHOOL.

It lasts *three* years (from 11 to 14 years of age) and is subdivided in two branches :

a. *Scuola Media : middle school* (classical branch) : this is intended to prepare children who will continue their education at higher levels.

b. *Scuola di avviamento : vocational school* : this is a professional-technical training school, which prepares children for practical professional activities : students of this school are not supposed to continue their education at higher levels.

III. SCUOLA TECNICA : TECHNICAL SCHOOL. It lasts *two* years (from 14 to 16 years of age); this is intended to complete and diversify the professional training (arts and crafts) already received in the vocational school.

IV. SCUOLA SECONDARIA SUPERIORE : HIGHER SECONDARY SCHOOL.

This comprehends two different branches, classical and technical, each represented by various types of schools :

a. Classical : (1) *Ginnasio-Liceo classico : Classical Gymnasium and Lyceum.*

It lasts *five* years (from 14 to 19 years of age) and prepares for all university departments (Facolta—Faculties). Classical (i.e. Greek-Roman) culture, philosophy and literature are particularly stressed.

(2) *Liceo scientifico : Scientific Lyceum.* It lasts *five* years (from 14 to 19 years of age) and prepares for some university departments (all, except Letters and Philosophy, Law, Medicine) : the accent is put on Science, however against a classical background.

(3) *Liceo artistico : Lyceum of Art.* It lasts *five* years (from 14 to 19 years of age) and prepares for the Department of Architecture. Art is particularly stressed.

(4) *Liceo (Istituto) Magistrale : Teachers' College.* It lasts *four* years (from 14 to 18 years of age) and prepares the future teachers of elementary schools. It may give access to a few

University Departments (Letters, Philosophy, Pedagogy).

b. Technical : This comprehends various *Istituti Tecnici Superiori* : *Higher Technical Institutes* with different aims :

(1) commerce (or trade-school for accountants)

(2) for surveyors

(3) industrial

(4) agriculture

They all last *five* years (from 14 to 19) and they do not give access to the university.

V. UNIVERSITA : UNIVERSITY. This is divided in various departments (faculties) and leads to a Doctor's degree. In general, it lasts *four* years, except for engineering (*five* years) and medicine (*six* years).

VI. POST-UNVERSITY INSTITUTIONS. These are intended to give a further specialization in a definite professional line. They last one or more years ; sometimes they are attached to university departments, sometimes they are separate institutions (like the Institute for Advanced Mathematics in Rome, the Institute for High Frequencies in Turin, and many others).

The connections between schools at one level and the next (as well as the various university departments) are shown in Table I.

2. THE MINISTRY OF PUBLIC INSTRUCTION. ITS ROLE.

The educational system in Italy is governed by a Ministry of Public Instruction : the Minister is a Cabinet member and reports yearly to the Chambers (Parliament and Senate) for the approval of the budget.

The Ministry comprehends seven general directions (or divisions) : for elementary education, secondary technical education, secondary classical education, universities, fine arts, museums and libraries, cultural relations with foreign countries, administration, and is

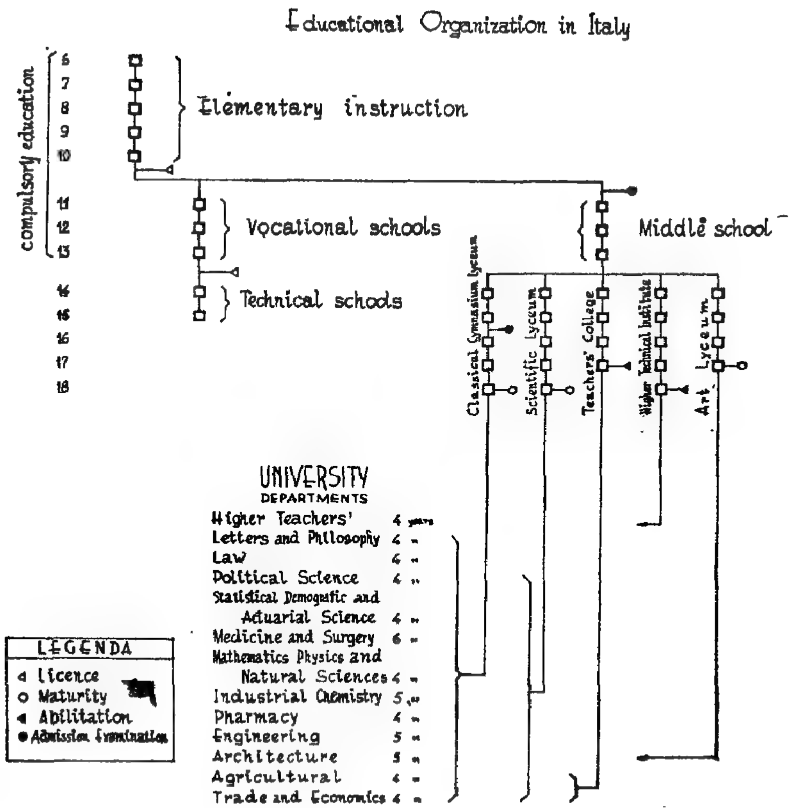
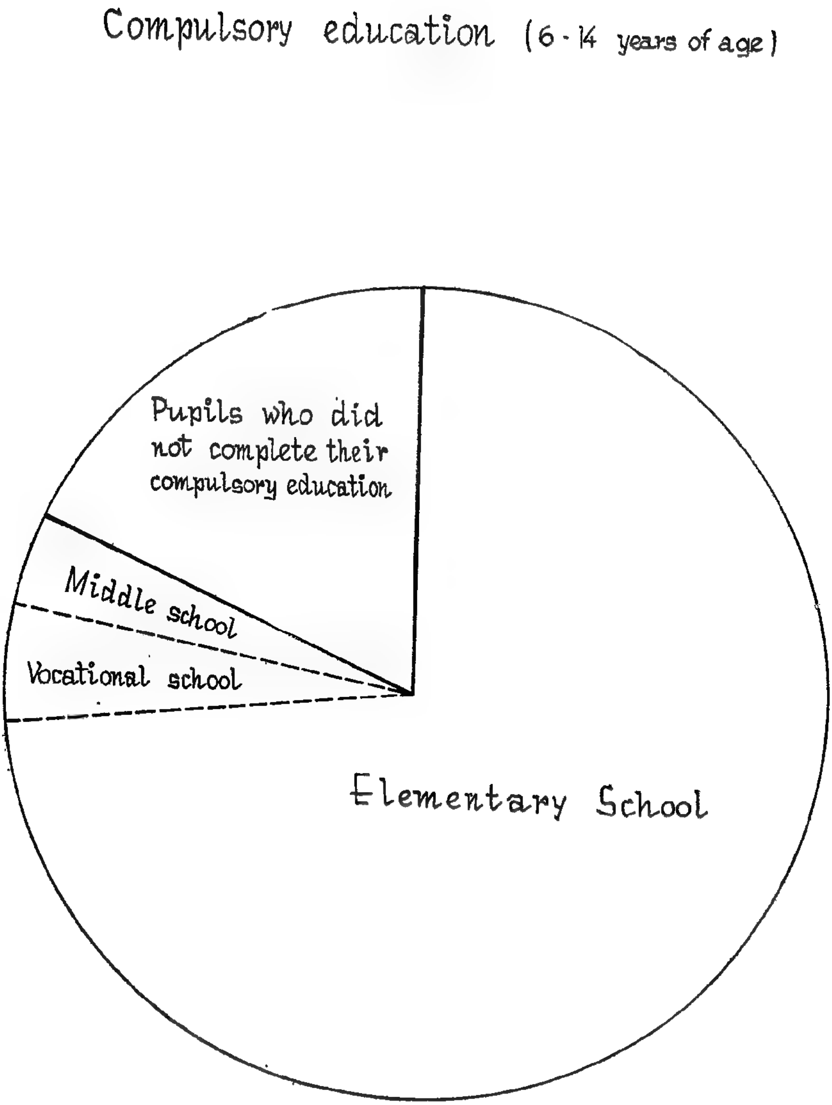

Educational Organization in Italy
compulsory education
6
7
8
9
10
11
12
13
14
15
16
17
18
Elementary instruction
Vocational schools
Middle school
Technical schools
Classical Gymnasium lyceum
Scientific Lyceum
Teachers' College
Higher Technical Institute
Art Lyceum
UNIVERSITY
DEPARTMENTS
Higher Teachers' 4 years
Letters and Philosophy 4
Law 4
Political Science 4
Statistical Demografic and Actuarial Science 4
Medicine and Surgery 6
Mathematics Physics and Natural Sciences 4
Industrial Chemistry 5
Pharmacy 4
Engineering 5
Architecture 5
Agricultural 4
Trade and Economics 4
LEGENDA
Licence
Maturity
Abilitation
Admission Examination

TABLE I


Compulsory education (6 - 14 years of age)
Pupils who did not complete their compulsory education
Middle school
Vocational school
Elementary School

assisted by the Higher Council of Education divided in different sections (elementary, secondary, university) whose members are teachers and professors elected by their colleagues of the same level (and a few appointed by the Minister).

The Ministry has a double function : it acts as an expert establishing the general standard of education at the different levels and as an administrator.

Schools (at various levels) are divided into two types : *state* (or government's) schools and *private* schools. These are subdivided into *legally recognized* schools and those simply *authorized.*

Each school is presided over by a Director or President ; all schools (except the universities) of a province (regional section of the State) are under the supervision of a (" provveditore ") representative of the Ministry in that province.

For the state schools the full financial burden for the teaching staff is assumed by the Ministry (other services are partly paid for by the province or local authorities).

Both on state and private secondary schools the Ministry exercises a supervision and a control on the didactic and disciplinary aspects using a selected body of " central inspectors " (and, for elementary schools, of " didactic inspectors " appointed on a national or regional scale).

The Ministry sets down for each type of school not a syllabus or teaching rules, but the amount of knowledge in the different subjects which must be possessed by the pupil at the end of each year, or at the end of 3, 4, 5 years depending on the type of school. This knowledge is ascertained by written and oral examinations which prove the " maturity " of the pupil at the end of the period and constitute either a " licence " from that type of school or " admission " to the next level (if any).

State schools and legally recognized schools award degrees or diplomas (of " maturity " or of " licence ") ; authorized schools

are completely free in their programme and do not grant degrees or diplomas : they must only satisfy some general requirements.

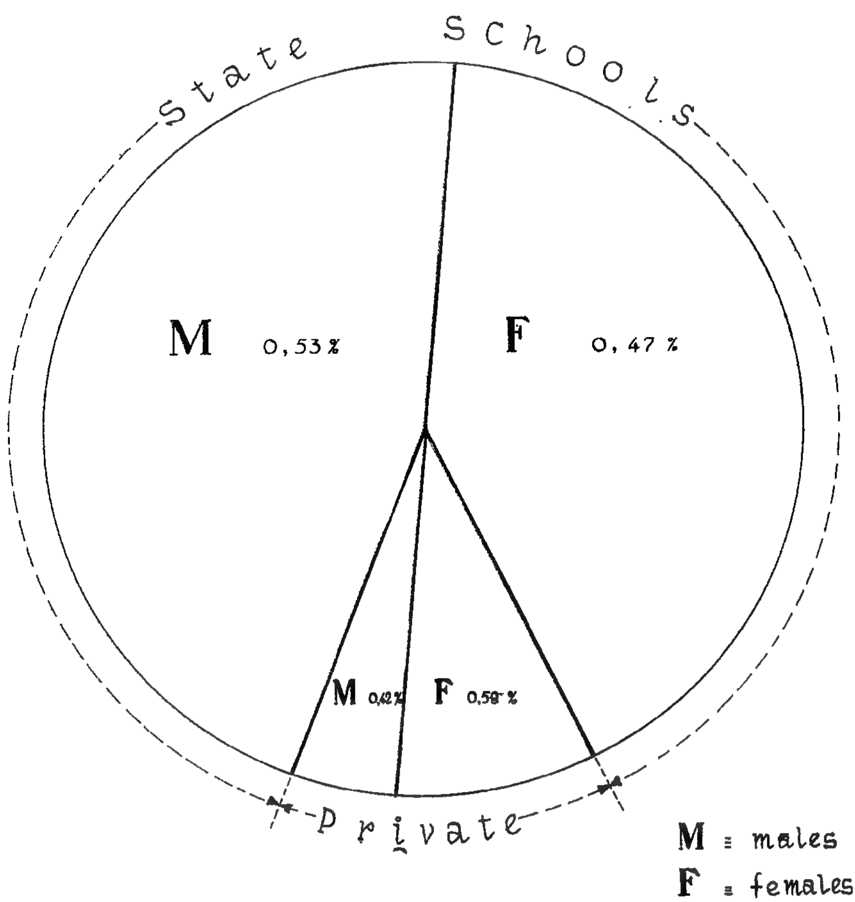
Another function of the Ministry is the appointment of new teachers and professors. This is done in different ways at different levels.

At the elementary school level the appointment of new teachers takes place on a *regional* scale. The representative of the Ministry in a province (provveditore) announces every second year the opening of a competition and appoints the members of a judging Commission (the slate must be approved by the Ministry). The Commission is composed of a President, a university professor or the Director of a secondary high school or a Central Inspector of the Ministry, and of members who are secondary school professors, and of one elementary school teacher. Candidates, who must have at least a diploma from the Teachers' College, must pass a definite set of examinations, designed to ascertain their cultural fitness and their didactic ability.

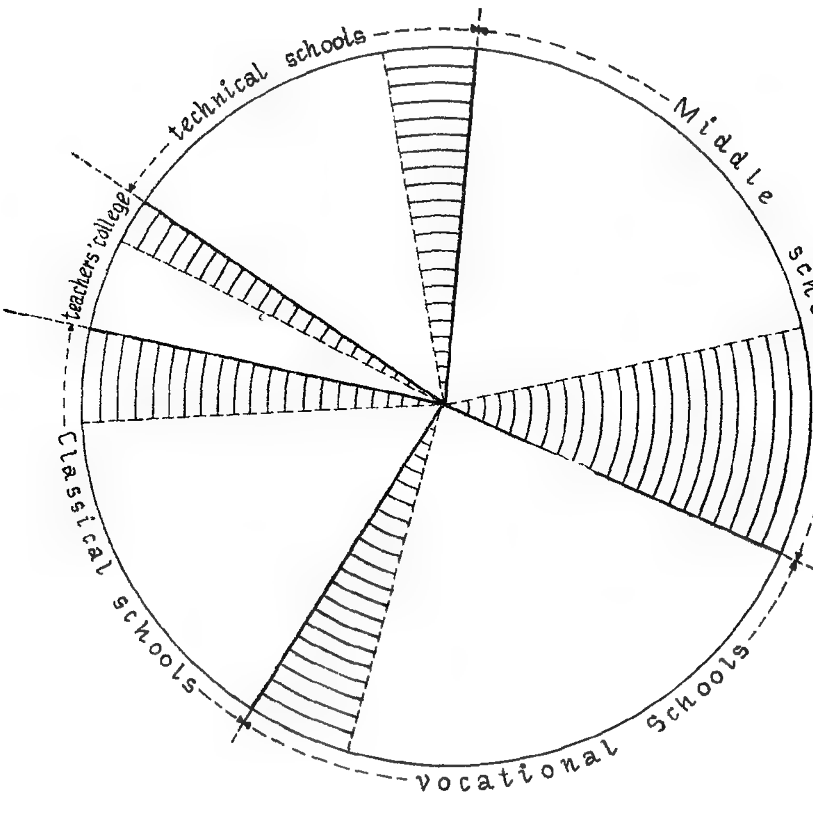
Competition for " didactic inspectors " (in elementary schools) is on a *national* (not regional) basis : candidates must be expert teachers with a certain number of years of actual teaching and with a diploma granted by the Higher Teachers Department (Facoltà di Magistero ; *three* university years).

Competitions for the appointment of secondary school teachers (at all levels) are also on a *national* basis. The competition (for each discipline) is announced by the Ministry of Public Instruction which also appoints the judging Commission, two thirds of the membership of which consist of university professors and one third of secondary school professors. Candidates must possess a Doctor's degree in the discipline for which they apply and they are rigorously selected through written and oral examinations which ascertain their cultural preparation and their didactic ability. Only those "habilitated " can enter on careers in the teaching profession : after three years of actual satisfactory teaching they are appointed full professors.

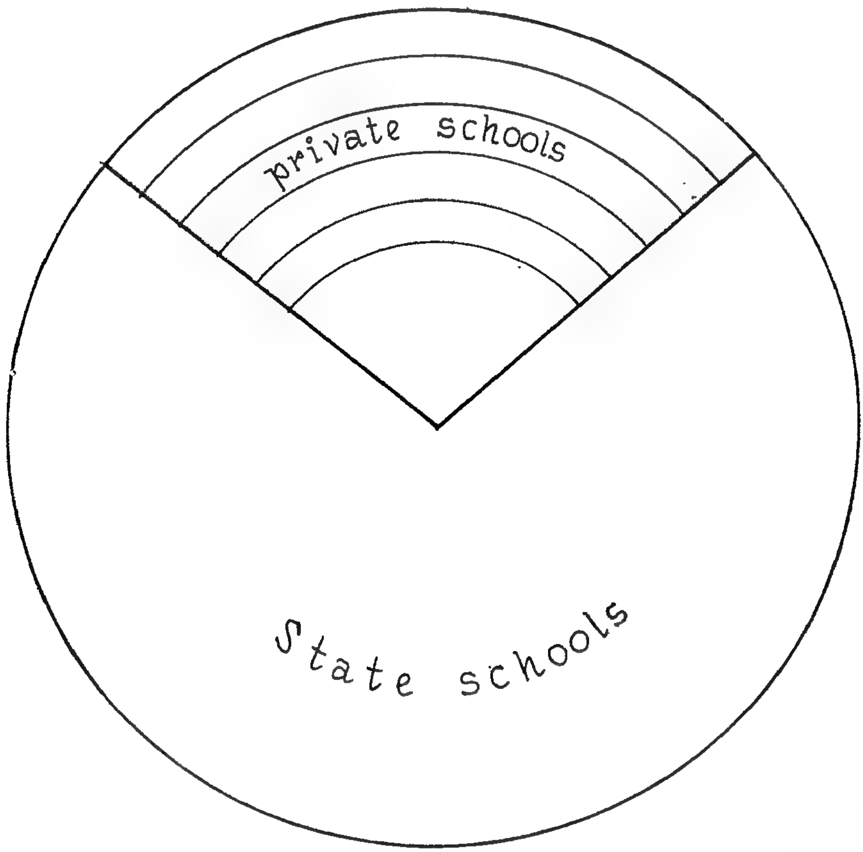
Elementary school attendance (1952-53)

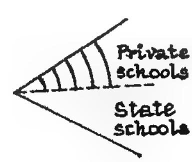


Pupils attending different types of secondary schools (1948-49)

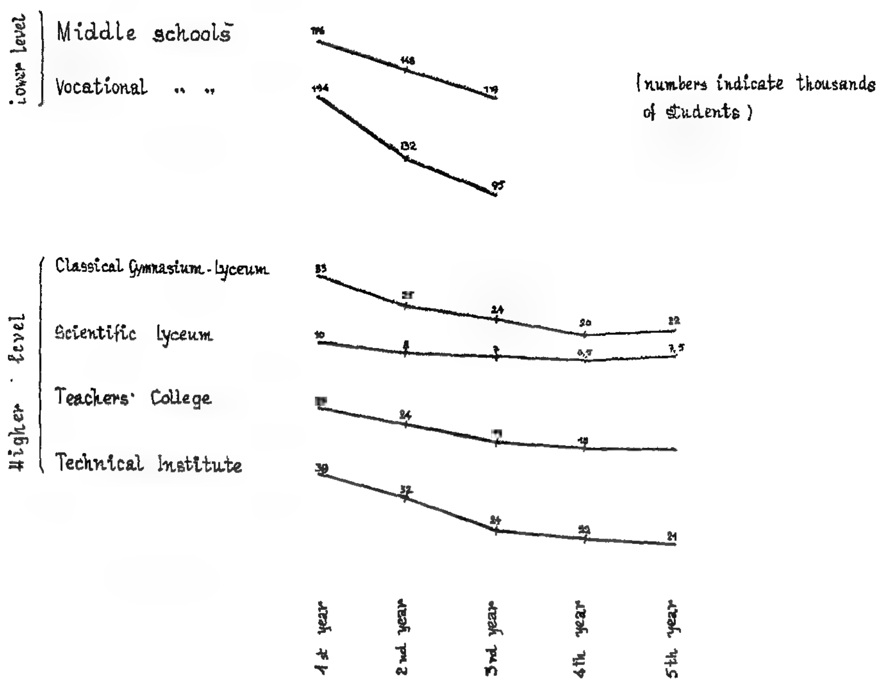


Distribution of pupils in State and private Secondary schools





Attendance of pupils in secondary schools
per year (1952-53)



Directors or Presidents of secondary schools are also selected by the Ministry on the basis of a *national* competition among full professors satisfying higher cultural and moral requirements than the average.

The selection of university professor is also the result of a *national* competition, of a quite different character.

When the Faculty of a certain department in a university decides to fill a vacant chair (all universities being self-governing bodies) it asks the Ministry to open a competition for that chair. If the application is approved by the Higher Council of Education (an elected body of university professors), the Ministry announces that the competition is open and at the same time invites the staff of the interested department in each Italian university to vote for two full professors in the required subject (for which the chair is vacant) as members of the judging Commission. The five full professors who have received the largest numbers of votes constitute the judging Commission.

Applicants are requested to send to the Commission, through the Ministry, their " titles ", i.e. their scientific publications, their curriculum vitae, a list of former appointments, and so on.

When the Commission meets, it examines carefully each document of each applicant and then compares the scientific merits of the various applicants : a record of this examination and of the subsequent comparison is later printed in the official bulletin of the Ministry (so that everyone can check the justification of the judgments). The Commission concludes its work by nominating, in order of merit, three winners of the competition ; only these can be appointed (even if the number of acceptable choices for a university chair are larger). The Faculty that asked for the competition may appoint any one of the three winners.

The system is so devised as to avoid personal, political or regional interference and to secure the choice of really the best candidate.

Three years after his first appointment, a university professor is subject to a new analysis of his scientific and didactic activities

by a Commission of his senior colleagues; if this analysis ends favourably he is promoted to a full professorship. Not even the Minister can remove him from this position (except in case of moral misdoings, in which case a disciplinary commission is set up to examine his case).

3. Compulsory education. Fees.

Education is compulsory in Italy for all children (boys and girls) from 6 to 14 years of age.

That covers the elementary and the lower secondary level. (Groups I, II in §1).

Therefore the elementary state schools and the vocational state schools are absolutely free of charge; the (lower) middle school (which is only a transition school giving access to higher levels) charges an annual fee which amounts practically to the cost of a package of cigarettes.

Actually the higher secondary schools also are practically free of charge: until a few years ago the annual tuition fee for these schools amounted to 70 cents (It. Lire 450); recently it has been raised to $3 (It. L. 2000) for the technical high schools and to $8 (It. Lire 5000) for the classical high schools. This increase is not intended to cover the government's educational expenses, but to provide financial reserves for fellowships to the most gifted children and for the maintenance of school buildings.

Annual fees for university education oscillate around $40 (somewhat higher when operating equipment is required); which is practically only one tenth (or less) of the actual cost of instruction (disregarding all other expenses). It follows that the State is actually supporting by far the largest part of the financial burden of education: this is a necessary consequence of the average economic conditions of Italian families. Nobody, or almost nobody, could afford to pay fees of the order of magnitude exacted in some other countries.

It must also be added that attendance at university lectures is open to anybody. There are no prerequisites laid down, and the

instruction is completely free of charge, provided the attendant does not ask for any certificate or diploma.

4. Historical survey of the preceding educational systems.

I have given in the preceding sections, a sketch of the present educational system in Italy. To understand how it was arrived at, it is necessary to take a glance at its historical background, at its successive developments and changes.

A century ago Italy was still divided into many small states, some of them under foreign domination. Unification of Italy was accomplished only in 1870 with the occupation of Rome (which had been proclaimed the capital town of the new nation already in 1861).

The year 1859 was particularly rich in political and military events : the war of liberation of the northern part of Italy from Austrian domination was actually started.

The fundamental law of public instruction in Italy, known as the " Casati Act " goes back to the same year.

Senator Casati, who was already a prominent political leader, was appointed Minister of Education while participating in the war. In the six months he remained in this capacity he fathered an excellent law which remained the corner-stone of public education in Italy.

The importance of the Casati Act is fundamental in many respects : it stressed the concept of freedom of teaching, the universal duty and right to elementary education, common to all social classes of people irrespective of their ability to continue their studies to a higher level, in a really democratic spirit. Elementary education became therefore compulsory and state schools were organized on a five year basis (divided in two periods or cycles) completely gratuitous. The State character of these schools not only removed from the elementary education the influence of particular groups, made them accessible to everybody and erased

all social class distinctions but was largely instrumental in bringing about the spiritual unification of Italy, giving a common pattern of education to all Italian regions.

Not less revolutionary was the Casati Act in the field of secondary education. This was at that time, in Italy as well as in other countries, restricted to a limited number of people, run almost exclusively by religious organizations, and was restricted to the study of the humanities in a classical sense. The Casati Act reduced the old Ginnasio from six to five years and added three years of Liceo (extending the secondary education to 8 years). It continued the study of Latin and Greek, but brought in philosophy (which was formerly confined to the university) and the exact sciences (mathematics, physics, natural sciences). The Ginnasio-Liceo, which is still the best and most homogeneous secondary educational system (and almost unchanged since the Casati Act) gives a solid cultural background and a complete picture of all intellectual activities so as to allow the student who is going to enter the university to make a conscious choice of the particular field in which his natural attitudes will be more suited and fruitful.

The Casati Act also provided a professional education for those who did not intend to continue their studies at the university level, creating the technical schools and institutes with clearly stated professional aims.

Later adjustments (1904) extended the compulsory education period from five to six years (between 6 and 12 years of age) and brought about other minor changes.

The next important step, after the Casati Act, was taken by the philosopher Giovanni Gentile in 1923 (at that time Minister of Education in the Fascist Government). He arrived at his reform—called the Gentile Reform Act—in accordance with his philosophic ideas and his large experience as thinker and educator.

It was the merit of the Gentile Reform to extend the compulsory education to 8 years (from 6 to 14 years of age). To this effect

"integration classes" (3 years, which later (1929) developed into the vocational and professional training schools) were added to the 5-year elementary classes (3 years of *pre-elementary* or nursery school were also added, before the elementary course, but were not compulsory).

One of the aims of the Gentile reform of secondary education was to reduce the number of State schools, in order to give them a higher standard and to encourage private initiative and competition: however, this aim was largely frustrated both by the pressure of an ever increasing population and the successive totalitarian trend of the Fascist Government.

Another important change introduced by the Gentile Reform Act was the suppression of the old technical school and the creation of a Technical Institute with an openly declared professional character and a Scientific Lyceum, of a cultural character, with emphasis put on Sciences and modern languages.

A logical consequence of the extension (to 8 years) of the compulsory education and a further step in the process of socialization of the secondary school was the creation of the *Scuola Media unica*: the *single Middle School* (similar to the *ecole unique* in France and the *Einheitsschule* in Germany) covering three years of work (Bottai Act, 1940). This single type of middle school prepares the youngsters for the four types of higher level: Ginnasio-Liceo (2 + 3 years), Liceo Scientifico (5 years), Lyceum of Art (5 years), Teacher's College (4 years).

The Middle School together with the Ginnasio-Liceo covers the same ground as the Ginnasio-Liceo of the Casati Act which is the traditional Italian school giving access to the university.

The gradual application of the Bottai Act was hindered by war and political events. The constitutional changes which happened after the war and the renewal of the democratic spirit prompted the necessity of a new reform of the educational system. This reform has been prepared by the Ministry, with the cooperation

of a large number of teachers, professors and experts, in the last years : but is still to be approved by the Parliament.

Programmes, except for minor changes (not affecting mathematics), are still those of the pre-war period. An exception must be made for the elementary school, in which new programmes have been adopted this year with a new subdivision in cycles (2 + 3 years).

Part II

Mathematical instruction in Italy

1. Mathematical instruction at the elementary level.
2. Mathematical instruction in Vocational and Technical Schools (lower secondary level).
3. Mathematical instruction in the Middle School (lower secondary level).
4. Mathematical instruction in the Classical Gymnasium-Lyceum (higher secondary level).
5. Mathematical instruction in other types of Lyceums (higher secondary level).
6. Mathematical instruction in Technical Institutes (higher secondary level).
7. Mathematical trends and their impact on textbooks used at the secondary level.
8. Mathematical instruction at the University level.

1. Mathematical instruction at the elementary level.

As already noted the elementary five-year course is divided in two cycles (2 + 3).

In the first cycle children begin their acquaintance with integers (from 1 to 20 and then from 21 to 100), with the writing and reading

of integers and the four operations on them (always avoiding remainders in the division).

The Pythagorean table is constructed and memorized.

Of course the teaching of numbers and operations on them is not done in an abstract way : the teacher must use every opportunity to show their necessity, following from very simple (one-operational) problems arising in their games or in practical life.

The intuition of the space is derived from direct observation of the most common objects (desks, chairs, cubes, spheres) and attention is drawn to some plane elementary figures (squares, triangles, circles) possibly showing their relations to solid objects.

In the second cycle the teaching of mathematics, although always connected with the other subjects of the cycle and always drawing its suggestions from practical problems, gradually differentiates itself and acquires a certain amount of autonomy. Operations are extended to integers greater than 100 and to decimal numbers (with no more than three digits). The idea of fraction is given (avoiding operations on them). Particular stress is put on mental operations even approximate. The metric system is also taught avoiding the use of less common units of measure. Children are drilled in recognizing the kind of operation required in one-operational problems : at the end of the cycle they must be able to solve problems involving at most three simple operations (exceptionally four).

In geometry, children must be able to calculate perimeters and areas of simple geometric figures (squares, rectangles, triangles, circles, up to regular polygons) ; and the volumes of very simple solids.

They are also encouraged to suggest, formulate and solve practical problems drawn from their own experience.

2. Mathematical instruction in Vocational and Technical Schools (lower secondary level).

Access is given to the vocational school by a " licence " from the elementary school.

It is intended to give notions of arithmetic and geometry necessary to practical life, particularly in the branch of activity (arts and crafts) children intend to develop.

The instruction comprehends arithmetic and geometry : to it 9 (weekly) hours are devoted (4 + 3 + 2).

The first year comprehends only arithmetic : children are drilled again in oral and written exercises on the four operations so as to insure a full command of them. Powers with integer exponents are then introduced both for integer and decimal numbers. In the arithmetic of integers the concept of prime number is introduced, the factorization of an integer in prime numbers and the research of the g.c.d. and l.c.m. are performed. Fractions and operations on them are also part of the first year programme.

In the second year the rule to extract the square root of a number is given and children are drilled in the use of numerical tables. Letters are introduced to represent numbers ; simple equations of the first order with one indeterminate are solved. Non-decimal systems are taught for measure of time and angles.

The second year programme includes also geometry : segments, angles and their measures, perpendicular and parallel lines, polygons (in particular regular polygons and their properties) circumferences and circles, their arcs, chords, angles on the circumference and their measures are part of it. Properties of plane figures with respect to movements and the theory of equivalence are also given.

In the third year the programme of arithmetic comprehends the theory of proportions, interest investment, discount and mixture problems ; in geometry proportionality and similitude (Thales theorem), and also measure problems, measure of areas and volumes of parallelopipeds, pyramids, cones, spheres, are developed.

Teaching of mathematics in vocational schools is mainly on a practical intuitive basis ; it is however advised not to discard completely the deductive method when experiment can only be used for checking results.

The technical schools, to which access is given by a licence examination from the vocational school, are largely differentiated in accordance with the various activities they are intended to prepare the student for (mechanics, engineers, electricians, radio experts, traders, agrarians, and so on). Also, the time devoted to mathematics differs from type to type, according to the special needs. The principal aim is to give the students a real command in the use of all notions already apprehended, in particular the units of measures, to be used in allied subjects (mechanics, physics, electrotechnics) ; large use is made of numerical tables, logarithms, slide rules, cartesian co-ordinates, graphs, circular functions, trigonometry of the right-angled triangle.

3. Mathematical instruction in the Middle School (lower secondary level).

The (single) Middle School, as was previously said, is intended for those children who will continue their studies at a higher (university) level. Hence, admission is restricted by higher requirements.

Having completed their elementary classes, applicants must pass an admission examination : the judging Commission is composed of secondary school professors assisted by an elementary teacher. The examination as far as mathematics is concerned, consists of a written test (solution of a problem implying three operations at most) and of an oral test in which the candidate must show his fitness for more advanced studies.

The teaching of mathematics in the Middle School has two aims : (a) to increase the amount of mathematical knowledge already acquired in the Elementary School ; (b) to prepare the youngsters for the study of rational mathematics and abstract thinking.

To accomplish these, although intuitive-empirical help must not be banned, the student is required to be precise in his concepts and their verbal expression. Teachers are asked to couple their teaching with some general mental operations, like ordering, partitioning, assembling, proportioning, equating.

The time devoted to mathematics is 9 (weekly) hours (3+3+3).

In the first year, operations on integers are reconsidered in order to put in evidence their formal properties (reflexive, associative, commutative, distributive laws) and the reasons for the practical rules already learned in the elementary school. The use of parentheses is emphasized. Then new concepts are introduced : the power (to an integer exponent) of a number, the prime numbers, the factorization of an integer in prime numbers, the research of common divisors and multiples, in particular of g.c.d. and of l.c.m. of two or more integers (also by the Euclidean process of successive divisions), the fractions and the operations on them.

Geometry in the first year of the Middle School is mainly intuitive, in order to introduce the knowledge of geometrical figures to be studied rationally on the next level : segments, lines, angles, perpendicularity and parallelism of lines, triangles, parallelograms and so on.

In the second year, after completing the operations on fractions, decimal numbers are considered (limited or periodical and their generating fractions) ; the rule to extract the square root and the theory of proportions also belong to this year. In geometry, properties of elementary geometric figures (including Pythagoras theorem, the rectification and quadrature of the circle) are given mainly in an intuitive way, but not discarding simple deductions as an inducement to mathematical thinking.

In the third year, relative numbers are introduced and operations on them. Here also begins algebra, with the use of letters, introduction of monomials (and their similarity), of polynomials, of the three operations (except division) on them, factorization in simple cases and remarkable identities. In geometry, circumference and circle and their measures are considered again not simply giving the rules for their calculation but indicating the geometric procedures which lead to them (inscribed and circumscribed regular polygons). A large number of applications of the elements of algebra

to geometric problems (particularly on the Pythagoras theorem) is suggested.

In solid geometry, relative positions of lines, of lines and planes, are also part of the syllabus.

Historical information about outstanding mathematicians (like Euclid, Pythagoras, Thales) is advised (but not compulsory).

4. MATHEMATICAL INSTRUCTION IN THE CLASSICAL GYMNASIUM-LYCEUM (HIGHER SECONDARY LEVEL).

Students enter the higher secondary level (Lyceums and Technical Institutes) after having passed their licence examinations from the Middle Schools.

I shall give in this section a sketch of mathematical instruction in the traditional Italian classical high-school : the Gymnasium-Lyceum. In the next section I shall compare the other types of Lyceums with this one.

Teaching of mathematics in the Classical Gymnasium-Lyceum has as its main purpose the developing and disciplining of the natural mental gifts of the student. It is not the material covered which is of paramount importance, but the method of acquisition of the results. Mathematics is not considered as a goal in itself, but a powerful instrument for the refinement of thinking in general : it is inducive to creative thinking, and also to clear cut ideas, to their exact verbal formulation ; it establishes the habit of controlling one's own process of thinking and of intellectual honesty.

Besides that, mathematics and particularly geometry is an integral part of the Greek (classical, in a Western sense) culture which would be badly mutilated and misrepresented if mathematics in the rational form should not appear in the syllabus of a Classical Lyceum.

As to the method of teaching : it must not be a purely logical exercise based on a certain set of postulates, but these must spring out of commonsense experiment and intuition retracing their

historical and psychological origin: the raw materials already assimilated in the lower level are subjected to a logical re-examination to find out the first principles (postulates) from which the same conclusions can be drawn.

The Gymnasium-Lyceum consists of 2 years of Gymnasium (which one enters with the licence of the Middle School) and 3 of Lyceum.

In the *Gymnasium* 4 (weekly) hours (2+2) are devoted to mathematics. Rational positive and negative numbers are introduced and operations on them, including powers with a relative integer exponent. Algebra includes polynomials and operations on them, basic products, factorization; algebraic fractions and operations on them, and the solution of equations of the first degree. Geometry starts with the usual elements (lines, half-lines, segments; planes, half-planes, triangles and polygons) and gives the criteria of equality of triangles, theorems on perpendicular and parallel lines, the sum of internal and external angles of a polygon, the triangular inequalities, properties of parallelograms in general and in particular cases; theorems on relative positions of lines and circles or of circles in a plane, angular properties of a circle; regular polygons and their properties, and some fundamental graphical problems. The theory of equivalence is developed for polygons up to and including Pythagoras theorem.

In the *first class of the Lyceum* (3 hours weekly), algebra is continued with the solution of systems (of no more than three) linear equations. The concept of real number is then introduced and operations on real numbers, including powers with integer or rational exponents (calculus of radicals). Then quadratic equations are discussed and solved, and systems of non-linear equations leading to resolution of quadratic equations are also solved.

In geometry the theory of proportions of geometric magnitudes is given; the similitude of triangles and polygons, the theory of measure and the area of polygons, are also developed rigorously.

In the *second class of the Lyceum* (2 hours weekly) the syllabus comprehends : the theory of arithmetic and geometric progressions, of exponential equations and logarithms, with numerous exercises and applications requiring the use of logarithmic tables.

In geometry : rectification of the circumference and quadrature of the circle are rigorously carried out. Solid geometry begins with notions of lines and planes and their relative positions ; orthogonality and parallelism, dihedra, trihedra and angoloids, polyhedra (in particular prisms and pyramids), cylinder, cone and sphere are studied.

In the *third class of the Lyceum* (2 hours weekly) geometry and trigonometry (with graphs of trigonometric functions, addition formulas and so on) are developed up to and including the resolution of triangles (Carnot theorem, Sines theorem and their applications).

In geometry the theory of equivalence of polyhedra and the rules to determine areas and volumes of other solids are studied.

In all classes numerous applications are made of algebra to geometry.

As to the teaching methods, or better lines of development and proofs, the teacher is absolutely free to choose those he prefers. For instance, real numbers are introduced either using contiguous classes and Dedekind sections or the decimal representation of numbers. Equivalence of polygons can be introduced either using decomposition in equal parts, or using the postulates for equi-extension of two figures (Severi).

Equivalence of polyhedra is developed either using the " Cavalieri principle " (equivalence of parallel plane sections) or, for prisms, the decomposition in equal parts, or finally, for pyramids, the method of " scaloids "

The theory of proportionality of geometric magnitudes is either reduced to numerical proportions, using their measures, or is given using the original Euclidean method (comparison of equi-multiples). And so on.

5. Mathematical instruction in other types of Lyceums (higher secondary level).

I shall report in this section on mathematical instruction in the other types of schools of classical character (Scientific Lyceum, Lyceum of Art, Teachers' College) referring to the Classical Lyceum and putting in evidence their differences with it. The character of the teaching is always the same (i.e. rational, and conceived as part of a general cultural frame), but more or less emphasis is put on mathematics with respect to other subjects.

In the five years of the Scientific Lyceum mathematics plays a major role. The time devoted to it is considerably larger than in the Classical Lyceum ($5+4+3+3+3 = 18$ weekly hours as against 13 in the last type) and so is the syllabus. The following subjects are to be added.

1. Elements of plane analytic geometry : Cartesian co-ordinates, graphical representation of functions of one variable (line, circumference, parabola with its symmetry axis parallel to one of the co-ordinate axes, hyperbola with the asymptotes parallel to the axes ; logarithmic curve and graphs of trigonometric functions).

2. Elements of infinitesimal analysis : limits, derivatives of elementary functions (with their geometric and physical interpretation) ; problems on maxima and minima ; definite and indefinite integrals.

3. Combinatory calculus : dispositions, permutations, combinations, Newton's formula for the expansion of the power of a binomial.

Particular stress is put on the discussion of the solutions of a quadratic equation whose coefficients depend on a parameter (interval of the variability of the solutions, their reality, in correspondence with the interval of variability of the parameter) ; geometrical problems relating to this situation are largely discussed. The comparison of the solutions of a quadratic equation with a real number is made by different methods (direct comparison ;

Cartesian rule and its extension; method of the fixed parabola; Tartinville's method).

As for the distribution of the subjects in different years, the first year in the Scientific Lyceum covers the same programme as the first two years of the Classical Lyceum (Gymnasium); analytic geometry starts in the second year and the elements of analysis in the fourth year.

In the *Art Lyceum*, the teaching of mathematics is coupled with that of physics and comprehends in all 17 hours (4+4+4+5). The syllabus is practically the same as in the Scientific Lyceum. It is to be said that owing to the particular aims of this kind of Lyceum, a lot of descriptive geometry (method of orthogonal projections, perspective and their applications) is given in the courses of "geometrical design" and "perspective" covering all together 15 hours weekly (4+3+4+4).

In the *Lyceum for Teachers* (or Teachers' College) the teaching of mathematics has a twofold purpose also: to promote the scientific culture of the pupils, and to give them the necessary professional fitness.

The teaching of mathematics takes up 12 hours weekly (5+3+2+2).

Algebra goes as far as the resolution of systems of linear equations and the theory of quadratic radicals.

Geometry covers the same ground as in the Classical Lyceum except for the fact that some subjects (real numbers, rectification and quadrature of the circle) are not developed in a fully rational manner.

On the other hand (because of the professional need for it) rational arithmetic is part of the syllabus.

6. Mathematical instruction in Technical Institutes (higher secondary level).

Technical Institutes, which in some cases do give access to a limited number of university departments, supply a highly qualified

and specialized technical preparation for various professional activities and are therefore largely differentiated.

The Technical Institutes prepare for the following main professional activities : Commercialists (administrative personnel) ; Land Surveyors ; Agrarians ; Nautical personnel (sea-captains, naval carpenters, engineers) ; Industrialists. These main categories (and particularly the last one) are sub-divided in numerous sub-sets (according to the various specializations) : the syllabus is highly differentiated (particularly in mathematics) to achieve the best preparation in the special field chosen by the student.

It would take too long to give a detailed account of mathematical subjects taught for the various categories of specialization.

It may suffice to say that mathematical instruction has a professional, not a cultural, aim. Financial and actuarial mathematics and calculus of probability have a paramount interest for commercial activities ; sea-captains need spherical trigonometry ; contractors must know not only accounting but also some of the mathematics used in engineering problems ; fine technicians must know how to measure the length of curves on a curved surface ; electricians, radio-technicians, opticians, designers need a different mathematical knowledge.

The number of hours devoted to mathematics is different in the various specializations and is generally confined to the first three or four years of the Technical Institute.

7. Mathematical trends and their impact on textbooks used at the secondary level.

Having given you a picture of mathematical instruction at the secondary level, and before passing to the analogous subject at the university level, it seems pertinent to say something about the textbooks used ; particularly to put in evidence the impact on them of the modern criticism on the foundations of geometry.

It will appear that outstanding mathematicians in Italy have always had a deep interest in secondary teaching and taken an

active part in writing textbooks, thus ensuring a high scientific level to secondary mathematical instruction (needless to say, there remain a large number of textbooks not satisfying this requirement).

At the beginning of the nineteenth century, mathematical textbooks were mere translations of foreign (particularly French) textbooks : Clairaut's *Eléments de Géométrie*, Legendre's *Eléments de Geometrie*, both remarkably different from Euclid's original Elements, were widely spread in Italian translations so were the *Leçons nouvelles de Géometrie élementaire* by Amiot (and Baltrer's *Elements of Mathematics*, translated by Cremona).

It was the merit of three outstanding mathematicians, Cremona, Betti and Brioschi to re-establish the teaching of geometry on the original Euclidean text (of which Betti and Brioschi gave an Italian translation in 1867). Having in mind pure scientific research at a higher level, Brioschi and Cremona stated that "our Gymnasiums and Lyceums must supply a highly elevated culture" and that their main purpose was "to teach the youngsters how to reason correctly, how to prove, how to make logical deductions ; therefore shortcuts and books which mix together geometry, arithmetic and algebra are no good ; Euclid is the real text which serves the stated purposes."

The same fundamental idea, that mathematics must be rigorous or it is not mathematics, was fostered by the analysis of the foundations of mathematics made by Dini, Veronese and Peano. This atmosphere of rigour created a peculiar character, and a permanent interest of university professors in the redaction of Italian textbooks at secondary level.

It immediately bore fruit : after the translation of Euclid by Betti and Brioschi, original textbooks, following Euclid's spirit but with a certain amount of autonomy and with new organizing criteria, were produced by Sannia, D'Ovidio, De Paolis and many others.

Of paramount importance in moulding the secondary teaching were " *The principles of geometry logically expounded* " by Peano (1889) and the " *Foundations of Geometry* " (1891) and the " *Elements of Geometry* " (1897) by Veronese. In these books, the purely abstract axiomatic-deductive method (completely severed from all physical suggestions) is strongly stressed. Veronese starts with the point (not with the space) and completely rejects the use of movements (as sometimes used in Euclid) assuming the equality of segments (or congruence of pairs of points) as a primitive notion, with no reference to its physical meaning. The same trend is followed in the analysis of the foundations of geometry by Fano, Pieri and Padoa.

This trend reached the peak of perfection in Hilbert's " *Grundlagen der Geometrie* " (1898).

It is the merit of F. Enriques, a mathematician combined with a philosopher and a historian, to bring, using the collaboration of U. Amaldi, the results of this critical movement to the secondary school level with their numerous textbooks.

Against the excesses of purely abstract axiomatic teaching at the secondary level, was the reaction of Peano (who insisted that giving a minimal set of postulates or proving every theorem was not necessary; but that only exact statements should be given) and the two philosophers and mathematicians G. Vailati and E. Rignano.

Enriques went a step farther: his philosophical mind was not satisfied with the purely formal perfection reached in geometry; therefore he started an investigation of the psychological ground of our assumptions (see his " Problems of Science "). On this ground he devoted a tremendous amount of work to didactical problems and to showing the impact of advanced theories on secondary mathematical instruction (following the example of F. Klein's " *Elementar Mathematik vom hoheren Standpunkte aus* "). He also developed a mathematical society (Mathesis) and founded a periodical concerned with these problems.

A larger appeal to intuition was made in the numerous textbooks of F. Severi : postulates are derived from everyday experience. He re-introduced the movement as a method of proof, however giving it a firm axiomatic foundation. He also gave many subjects (theory of parallels, of equivalence, proportionality of geometric magnitudes) a new and original treatment.

It is impossible to give here a similar analysis of textbooks in arithmetic and algebra ; suffice it to say that also in these fields the interest of university professors in secondary teaching has always produced excellent textbooks (Pincherle, Vivanti, Gazzaniga, Bortolotti, Tonelli, Cipolla, Nicoletti, Sansone).

Resulting from the same interest given by university professors to secondary teaching, are the volumes edited by F. Enriques on questions concerning elementary mathematics and the excellent encyclopaedia of elementary mathematics edited by L. Berzolari : practically all Italian university professors took part in these enterprises.

As a further proof of the continuous interest of university professors in secondary mathematical instruction, let me remind you that it was at the fourth International Congress of Mathematicians in Rome (1908) that the International Commission on Mathematical Instruction was born : the resolution, later adopted by the Congress, which was its act of birth, was put on the floor by my unforgettable teacher G. Castelnuovo ; among the first members of the Commission were G. Vailati, G. Castelnuovo, F. Enriques, G. Scorza.

8. Mathematical instruction at the university level.

The bulk of mathematical instruction at the university level is given in the Department (Facoltà) of Physics, Mathematics and Natural Sciences. Other mathematical courses are given in the Departments of Political Sciences, of Statistical, Demographic and Actuarial Sciences, of Economics, and so on. It would take too long and be of little interest to report on all these special courses ; but it would also be, in a certain sense, useless because, since the Gentile Reform Act, a student is free to take some courses also in

departments other than those of his specialization (e.g. a student of philosophy can take courses in mathematics or vice versa ; and this actually happens).

Let us confine ourselves therefore to the main mathematical courses. These are divided in two 2-year terms : the first biennium and the second biennium.

Students entering the first biennium must have a licence (after a maturity examination) either from the classical or the scientific Lyceum.

Mathematical courses in the first biennium are compulsory for students of engineering, of mathematics and of physics (chemists can take the same courses or alternatively follow analogous courses specially devised for them). The courses are labelled as follows :

1. Mathematical analysis (finite and infinitesimal) ; a two-year course (3 hours of theoretical lectures, 3 hours of exercises and applications, weekly).

2. Analytic, projective and descriptive geometry ; a two-year course (with the same schedule as above).

3. Rational mechanics with graphical statics ; this is a one-year course, given in the second year of the biennium (4 hours of theoretical lectures, 3 hours of applications).

I have purposely used the word " labelled " because the label does not actually indicate the composition of the course or its extension, nor the way it is developed. This is all the more true because courses with the same label differ widely in character from university to university. There is no syllabus defining a course : a skeleton indication is supplied by tradition, but the professor is absolutely free to cover more or less ground and to develop the subject the way he likes. This freedom of teaching finds its expression in the mimeographed (or printed) notes that practically every professor prepares for his students : only these notes define the course; they reflect the personality of the individual professor,

As an instance, a course in mathematical analysis contains the usual notions of algebra (including the theory of equations), of limits, of series of functions, differential and integral calculus (with multiple integrals), series developments, Laplace transforms and inverse transforms, theory of differential equations and systems of them, partial differential equations: but it may contain other subjects and in any case, different emphasis is put on these various subjects by different professors. Common to all of them however, is the emphasis on mathematical rigour.

The same is true of the geometry course : it certainly includes plane and solid analytic geometry, projective and descriptive geometry ; but projective geometry may be treated either synthetically or analytically.

The group idea is in some cases assumed as the fundamental idea of the treatment of the subject.

Besides that, this course contains many topics (singular points of curves or surfaces, linear systems of algebraic curves, Plucker formulas for an algebraic curve, differential study of curves and surfaces in the ordinary space up to and including the fundamental quadratic differential forms) which may be characterized as introductory to algebraic and differential geometry.

As mentioned above, these courses are compulsory for different groups of students (who, besides these, have other compulsory courses which differ for different groups). This fact ensures that mathematical instruction for engineers is on a pretty high level : a few topics however are added to the general syllabus, for prospective professional mathematicians.

Written and oral examinations are required at the end of each course.

After the first biennium, different groups have different curricula and we will follow only the students of mathematics. These continue their studies for a second biennium aiming at a Doctor's degree either in mathematical sciences, or in mathematics and physics.

Generally speaking, the first type of doctorate is chosen by those who want to continue in research work and scientific activities; the second type is devised for those who want to become professors of mathematics and physics at the secondary school level (at this level, since the Gentile Reform Act, the teaching of mathematics and physics is entrusted to the same teacher).

The curricula followed to attain either of the two types of doctorate are pretty similar : in the second type, mathematics and physics are equally balanced and attention is paid to teaching problems.

The curriculum of the second biennium for the Doctor's degree in mathematical sciences comprehends three compulsory subjects : higher analysis, higher geometry and mathematical physics, and three more courses (so-called complementary) which can be chosen by the student among the following : astronomy, numerical and graphical methods of calculation, theoretical physics, advanced physics, algebraic geometry, differential geometry, higher mathematics, theory of functions, complementary mathematics, advanced mechanics, theory of numbers, geodesy, topology, probability, calculus, actuarial mathematics, history of mathematics.

The curriculum of the second biennium for the Doctor's degree in mathematics and physics comprehends four compulsory subjects : advanced physics, higher analysis, theoretical physics, complementary mathematics, and three courses chosen among those already mentioned and many others relating to physics (electro-technique, technical physics, geophysics, mineralogy, spectroscopy, statistical mechanics, electromagnetic waves).

Each course consists of three hours weekly of lectures on theory (with exercises or laboratory work if required) : examinations at the end of the course are required.

Courses at this level are of monographic or research character and their content varies from year to year ; the teacher is completely free and is alone responsible for the choice of the topics, which are generally related to his own scientific work at the time. Labels are,

on purpose, very general and indeterminate. Higher analysis may mean a course on calculus of variations, or on operational calculus, or on partial differential equations and so on ; mathematical physics may mean *optics* or *elasticity* or *electromagnetism* and so on ; differential geometry may mean Riemannian geometry, theory of connections, theory of groups and many other subjects.

As a rule, no textbook is adopted (which would limit the originality of the course) ; sometimes mimeographed notes are prepared by the professor but generally students are required to take their own notes.

When the student has completed his examinations he must prepare a written dissertation on a subject he chooses freely (generally under the guidance of one of his professors) : the dissertation must show his ability to do scientific research (this requires, in many cases, an additional year).

When the dissertation is approved, the student applies for admission to the final doctorate examinations : these consist of a written and oral " cultural examination " (a kind of general survey of the candidate's knowledge in mathematics or in mathematics and physics) ; of an oral exposition of the written thesis and of three oral theses before a full commission of eleven professors.

If everything goes smoothly he is proclaimed a Doctor ; that is the end of his studies and there begins the competition for a teaching post or the fulfilment of scientific ambitions.

University of Rome

TYPOGRAPHY AND THE TEACHING OF MATHEMATICS

By T. A. A. BROADBENT

We are well accustomed to habitual sneers at the " narrow " specialist; yet the true specialist, who has slowly and with labour attained to a clear and profound understanding of one field of human activity, may be trusted as a rule to have at least a sympathetic insight into the general mode of thought of his contemporary workers in other domains. Thus may I be permitted to recommend to your attention the work of a distinguished French historian, Marc Bloch, entitled *Apologie pour l'Histoire*, available also in an English translation, and particularly to direct you to two passages, which I shall quote, in which he seems to me to have summed up, without perhaps being fully aware of it, the functions and character of mathematics :

> " ...each science has its appropriate aesthetics of language. Human actions are essentially very delicate phenomena... Properly to translate them into words and, hence, to fathom them rightly (for can anyone perfectly understand what he does not know how to express ?), great delicacy of language and precise shadings of verbal tone are necessary " :

and again

> " ...the first tool needed by any analysis is an appropriate language ; a language capable of describing the precise outlines of the facts, while preserving the necessary flexibility to adapt itself to further discoveries and, above all, a language which is neither vacillating nor ambiguous "

Mathematics is a phenomenon of human action, a phenomenon of the human mind, infinitely more complex and varied than those

This address was given at the South Asian Conference on Mathematical Education held on 22-28 February 1956 at the Tata Institute of Fundamental Research, Bombay.

of the human body; it requires great delicacy and precision, and is sterile unless it can be communicated from one mind to another, a communication which demands the perfect understanding, of which Bloch speaks, and therefore demands a mastery of expression. In fact, it would not be any great exaggeration to say that mathematics does not need but is a language, of exactly the type described by Bloch in his second passage—capable of precise descriptions, flexible enough to adapt itself to and even to point the way to further discoveries, sharp, definite and unambiguous. Of all languages, mathematics is best fitted for the communication of abstract rational thought; and by reason of its nature, it is a universal language—possibly the only really efficient universal language.

Further, mathematics is to a very large extent a written language, and, specially, very much a printed language. Many of us learn best through the eye; and we depend on print to supplement memory. The mathematician's workshop, his laboratory, is a library; pencil, paper, books are his test-tubes and Bunsen burners, his raw material and his re-agents.

But, if we are prepared to agree that mathematics is essentially a language, the universal language of abstract logical thinking, and that it is to a very large extent a printed language, then, two sets of implications deserve further study.

First, if mathematics is a language, then, it must be taught as a language, serving both for use and for enjoyment, and we must be prepared to study the methods whereby other languages are taught, and in particular, the methods used to teach a language which is not the mother-tongue of the pupils. The mathematical prodigy, of course, may require special treatment, but few teachers are likely to be so fortunate as to number many mathematical prodigies among their pupils; indeed, if we should be so fortunate, let us thank Heaven, and then refrain from attempting to thrust our teaching down the throat of one who does not require such infantile nourishment. My concern here is with the average pupil, whom we meet in large numbers.

We may, I think, discern four stages or ideals in the process of teaching a language. To read, to write, to speak, and finally, to think in that language without any need of mental translation; these are the aims: though often the fourth is beyond the power of the ordinary child, while the third may be so unless the child can live for some months in an environment where the language in question is the normal spoken tongue. If we are to train a pupil to be a professional mathematician, then all four stages must be achieved; to speak and to think mathematically are the hall-marks of the professional. But if mathematics is to remain a basic and prominent feature of the general school curriculum, then all, or almost all, our pupils must learn to read and write mathematically, so that ultimately they may both use and enjoy their mathematics. Excluding for obvious reasons the topic of elementary arithmetic, and envisaging the pupil who is embarked upon algebra and geometry, what can we learn from the teaching of languages at the corresponding stage? Well, first we may recall that it is really not so very many years ago since a pupil learning a language other than his mother-tongue began with grammar, with declensions and conjugations and continued with these until he could repeat them with parrot-like accuracy, even though he might not be able to ask for his dinner. Today we start with simple but significant sentences, describing situations, meeting needs, answering questions; but they are all within the compass of the child's experience. The corresponding change in the teaching of mathematics has also been achieved: no longer in algebra do we insist on hour after hour of pure manipulation,—multiplication, division, factors, simplification. It has often been argued at some length—and with some heat—that we ought to start with the formula, or that we ought to start with the problem; this seems to me a somewhat unrewarding controversy; what we have to start with are simple but significant sentences, describing situations and asking questions within the scope of the child's experience, and showing the child how he may himself discover the answers. At a later level, we no longer force the pupil through hundreds of manipulative examples of the technique of

differentiation and integration before allowing him to see that a mastery of this technique will enable him to pose and to answer questions of genuine importance not amenable to the more elementary techniques which he has already acquired. In doing this, whether we have copied the teacher of languages or he has copied us, at least we have gone far to free ourselves from what was the besetting sin of the teaching of mathematics and of the classical languages in the late 19th century. While one must naturally respect the high scholarship of the grammarians of that era, one may reasonably protest that to some of them grammar was no longer a means to an end but an end in itself, Latin and Greek were not only dead languages, but languages which had never been alive; to think of Plato and Archimedes, Livy and Virgil as human beings more or less like ourselves was not only not a help, it was almost a hindrance to the acquisition of pure scholarship. And mathematicians are not altogether free from a similar reproach ; the acquisition of a technique was an end in itself, not something to be used and enjoyed, but merely something to be acquired. That attitude has been abandoned, I hope for ever, and having set our feet on the right road we are now ready for a further progress. I have stressed that, to me, mathematics is a language to be used and *enjoyed* ; few would quarrel with the first part of the description; the second would arouse more criticism; yet surely it is as important as the first. How are we to set our pupils on the way to enjoying their mathematics?

Obviously, by giving them mathematics in an enjoyable form. I do not mean by this that we should attempt to introduce an element of comedy into the subject, that we should make great use of the comic strip, that we should conceal or ignore difficulties. No ; but the mathematics we offer, at whatever level, must be the best of its kind, must be as perfect in form as we can make it. It must be simple ; it need not be, it should not be necessarily childish. Again, we can learn something from our colleagues in the department of languages. Once the child has mastered the elements of reading, the wise teacher in these days will no longer confine the pupil to prose which is childish, though he will take care that

it is certainly simple. Prose which is intentionally and deliberately "written down" to a presumed child-level is much more likely to induce contempt from the reader than admiration. The teacher of English finds that it is well worth searching for prose or verse which is simple without being childish; he can find it, for example, in many places in the authorised version of the Bible, in most of the writings of John Bunyan, in parts of Defoe and Swift, in the verse of Chesterton, or Macaulay. Here the style has all the elements of simplicity; short sentences, a high proportion of monosyllables; the imagery concrete and familiar. Yet no one would deny that here we have English at a high level, still thoroughly suitable for the child reader. (I must apologise for taking my examples here from English, but it would clearly be impertinent of me to express similar opinions on writings in other tongues; I am sure that corresponding instances in French or German could readily be found.) I am not sure that we have learned the analogous lesson in mathematics. We have not yet solved the problem of presenting genuine mathematics in a sufficiently simple form.

May I go further? Speaking from experience as an editor for some 25 years, and being careful to preface my comment with the restriction that I can speak only about mathematics in English, I would say that the cardinal fault of most mathematical exposition, from the elementary school textbook up to the advanced treatise or the research paper, is carelessness of style. Please do not misunderstand. I am not asserting that style is as important as content; far from it. Accuracy of content must come far above any other requirement. But then, much care is always paid to this requirement by the average author. There are mistakes, of course, but these are seldom the consequences of negligence. In any event, papers which contain too many mistakes in content are unlikely to be published anyway. But accuracy of content, though of over-riding, primary importance, is not quite all. A completely solipsist philosophy is not common among mathematicians; we, most of us, regard it as a duty and to some extent a pleasure to communicate our discoveries to others. Yet far too often we take very little care

about this side of our work. It is natural, indeed, that there should be a considerable decrease in tension once the creative idea has emerged and proved itself of value ; writing out a fair copy for the press can be sadly tedious. But the tedium is no excuse for negligence. Kelvin called Fourier's *Theory of heat* a mathematical poem; this perhaps is too exalted an aim. But we can all write good, simple mathematical language if we try hard enough ; and if we do not try hard enough we are not doing our full duty to our colleagues and our subject, to which we should be, in Bacon's phrase, a help and an ornament.

There is one golden rule : the laws which govern the construction of mathematical prose are precisely those which govern the construction of good non-mathematical prose. A sentence must contain a principal verb, there must be agreement in number between subject and verb, a transitive verb requires a direct object, and so on. This of course is mere journeyman stuff, too often neglected, but still only the crudest and most elementary requirement. The author who wishes to make his paper as clear as possible must go further; he must first weigh and digest the words of A. E. H. Love, stressing the need for training in the means of expression, the need for the patience which will re-write and if need be re-write again in order to produce a mathematically articulate exposition; then he must go beyond the crudest forms of correctness and pay attention to style, even to delicate shades of style, coming back again and yet again to the fundamental principle that what is not good prose cannot be good mathematical prose. He must scrutinise carefully even the details of punctuation.

Consider an example:

"Let $f(z) = \Sigma\, a_n z^n$ be a function which is regular in the unit circle."

What is wrong with this? Nothing very much, but it is the thin end of a wedge whose broader part would admit

"Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ be a function which is regular in the unit circle."

Now it is true that the excellent pamphlet on writing mathematics for the press issued by the London Mathematical Society says that the equality sign is to be treated as a fully inflected verb, but in ordinary prose the inflection is often visible in the form, whereas the sign of equality cannot show of itself how it is being used.

Consider a parallel in ordinary prose:

"Then Mr. Blank, the Prime Minister, said in reply..."
or

"Here Mr. Dash, who is the leader of the party, rose to point out...".

In the first of these, we should certainly never omit the commas, in the second a slovenly writer might allow himself to omit them, but no writer with any pretensions to precision would ever dispense with them. The reason is clear: in each example we have a clause in apposition, and this must be kept separate from the main structure, such separation being best indicated in normal prose by enclosing the phrase in commas. In the mathematical example, it is $f(z)$ which is regular, with the series as its representation, so if we are strictly to follow the principle, we should enclose the clause in apposition, namely "$= a_0 + a_1z + a_2z^2 + \ldots$" in commas. This in a displayed formula is decidedly unconventional, yet to leave the sentence as it stands is to subject the reader to an unnecessary strain. Why should this be so? Because the writer has fallen into the cardinal error of terse style, confounding two ideas into a single sentence. We should be prepared to say "Let $f(z)$ be a function regular in the unit circle, having the expansion..." or "Let $f(z)$, whose expansion is..., be a function regular in the unit circle". Punctuation is essential to ready understanding, and its rules are clear and simple; but they cannot always be directly applied to mathematical prose, and, therefore, when they cannot be so applied, the sentence should be completely re-cast.

Good structure requires not only crisp, precise sentences; these sentences must themselves build up into paragraphs. Many authors have no care for the importance of the paragraph. Some write on

and on until they become tired of seeing a monotonously even left-hand margin, and in desperation they make a break which may well be between two closely related sentences. Others are so alive to the need for a break that they rarely allow more than one sentence to a paragraph. I wish that all mathematical writers using English could be compelled to study the structure of Macaulay's prose, particularly in his *History of England*. I am not to be taken as endorsing his opinions. But there is not one obscure sentence in the whole set of volumes; and each paragraph is constructed with the utmost skill and craftsmanship; each has its own main topic, yet each forms a link in a continuous chain of description or argument.

Let me take a specific example. For definiteness, let us suppose that we are Picard, about to write out for the press a proof of the new result that an integral function which does not take two specified values is a constant. How should we divide our paper into paragraphs? We want to establish that an integral function which never takes the values a, b is a constant. First, then, we shall use one paragraph to show that a linear transformation allows us to take the special values $a=0$, $b=1$. Next, if ν is the function inverse to the modular function, we have to show that $\nu(f)$ is an integral function with positive imaginary part; this, to Picard, is not a classical result, so it must be proved. It makes a second paragraph, which may of course be set out as a lemma; that is, a self-contained result, not particularly interesting for its own sake, but essential to the argument. This is a reasonable usage of the word "lemma"; the recent tendency to strive for the maximum lemma-density per paper is not a fashion to be encouraged. Finally, from the lemma it follows that $\exp\{i\nu(f)\}$ is a bounded integral function and is therefore by Liouville's theorem a constant. Here is our third paragraph. In fact, a little thought shows that the whole proof falls easily into three main stages, and these stages show the natural paragraph breaks.

So far, we have been concerned mainly with the relation between the writer and the reader; but there is usually a middleman. His

function, unlike that popularly attributed to middlemen, of sitting back and doing nothing save collecting the profits, is of prime importance. The printer is the channel of communication from author to reader, and if the co-operation required by the reader from the printer is to be adequately received, then the co-operation between the author and the printer must be close and intelligent. But it must be a co-operation. Printers and compositors are highly intelligent, men of skill and craftsmanship, but their powers should not be over-estimated. Any editor here present will recollect the feeling of irritation and dismay which arises whenever he has received a covering letter to say : " I am afraid that my manuscript is not very carefully written, and may need some revision, but no doubt the printer can attend to all that"! Quite often he cannot, and even if he can there is no reason why he should—it is not his job. His job can be simply and even adequately described by saying that he is bound to reproduce what is in the script. With some printing houses, where house rules are sacrosanct and inviolable, the compositor will adapt symbols to meet the needs or imperfections of the manuscript, but even in these circumstances he can only repair, he cannot add nor should he subtract. Thus between the reasonable and legitimate constructions of the compositor and the intentions of the writer, there may easily be a gap, and if the writer is careless this gap may be a wide one. Such a defect in co-operation may diminish substantially the visual impact of the text on the reader. For many readers, this visual impact is of prime importance ; perhaps some of us here would concede under pressure that part at any rate of our mathematical knowledge was gathered and is retained by a vivid recollection of the appearance of some printed page. Nor need we be too much ashamed at such an admission, since Rayleigh's *Life of Sir J. J. Thomson* shows that much of J.J.'s phenomenal knowledge of the literature of theoretical physics was gathered and retained in this way. Whether psychologically this is a good or a bad thing, I do not know; what matters is that it is a widespread fact that visual impact is all-important for many students of mathematics.

Since the role of the compositor is essentially limited to the reproduction of what is in the script, the onus for visual impact falls heavily on the author. While this is true for all levels of mathematical exposition, it has a particular importance for authors of textbooks, more especially for authors of first courses, whether these be first courses in arithmetic or algebra or calculus or any other topic. The author's task is not at an end when he has collected his material, arranged it in order, written it out accurately and clearly, and checked the substance of the completed manuscript. If he is wise, he will now visualise as clearly as he can the probable appearance of the pages of his work in print. To do this, he must first know something of the restrictions imposed by cold rigid metal types. Not everything the pen performs can be reproduced typographically; some things are impossible. And it is not without value to remember that what is possible may be expensive, and the author concerned with the mind of the student may be recommended to spare an occasional thought for his pocket. Secondly, he must endeavour to see the printed page as a whole and to keep it at least in his mind's eye while writing up his manuscript. There is a well-known English textbook, now some fifty years old and so out-moded and antiquated, not indeed free from gross errors, which still sells its thousands of copies every year and is used widely throughout the Commonwealth. Why? Not simply reverence for the antique. The author, as I have good reason to know, could envisage precisely the form of printed page which he desired to achieve, he knew exactly what would go on one line, how much manuscript would exactly fill one page of print, so that awkward page-turnings in the middle of an important argument could be avoided, he knew just how much and how little display to give to symbolic matter, where to give space, when to use Clarendon and italic types. Thus though today we have many better books available on this topic, the veteran holds its place, not on its mathematical merits, which are now much dimmed, but because the author had the gift of envisaging the visual impact of his work in print, or rather, because by painstaking thoroughness he had earned this gift.

Trivial, technical matters ? Yes, if you like. And so beneath the dignity of the mathematician, unworthy of his attention ? No, emphatically, no. One of the outstanding English treatises of my time, a work held in the utmost respect throughout the mathematical world, was made much more difficult to read than it need have been by lack of proper co-operation between author and printer, resulting in cramped pages, crowded masses of symbols, and unemphasised formulae. For the high-class professional, this perhaps does not matter much. But we must think of the average reader, in this case say a good mathematician not specialising in this field. His task is harder just because an undue proportion of his attention has been diverted to the preliminary task of filling the gap between author and printer, a gap which ought to be filled by the author, not by the reader. And if this diversion of attention occurs in a school textbook, the result can be disastrous : I know of a pupil working on his own who was driven almost to the point of despair and defeat merely because the author of the book he was using had never bothered to explain that exp x means the same thing as e^x. To say that he, the novice, ought to have inferred this identity from the context is simply the excuse made by the slovenly author.

Among the golden rules, then, are these : good mathematics must be written in good prose ; the author should realise the scope and limitations of metal type ; he should develop in himself the capacity to visualise what his manuscript will look like when it is transferred to print. Less than this, and he is likely to fail in his aim, his duty, of communicating his thoughts, in precise form and without loss of accuracy and substance, to his readers.

But this is not the whole duty of the author. He is to learn from the printer ; but very often the printer has much to learn from the author. At the lowest level, the author may have to teach the printer that, for instance, a Greek fount which does not clearly distinguish between a lower case italic a and a lower case 'alpha' is not suitable for mathematical work, or again, that an unfamiliar symbol should never be improvised (the attempt to construct an integral sign out of a pair of brackets is an extreme instance, but

has actually been seen). At a higher level, authors requiring new symbols or new groupings of symbols should consider the typographical side of their requirements, so that before a new symbol is launched on the scientific world its typographical difficulties and implications may have been examined.

Further, there is a field, if not of genuine research then of investigation, open to some one industrious enough and humble-minded enough to explore. How does the ready appreciation of printed matter increase with increase of age? Little enough has been done even on the general topic of the impact of print on the child-mind. For instance, a hundred years or so ago it appears to have been taken as an axiom that the younger the child, the smaller should be the fount size. Today we have gone to the other extreme, and hold that the smaller the child the larger the fount size. But this tendency has its dangers. I have seen a child capable of fluent reading hesitate lamentably over such a simple word as "spaceship"; inquiry showed that the child could read this at sight in a normal size of type, but failed to recognise it in the large type in which it was in this instance presented, because the word had spread out beyond the physical capacity of the child's eye. Those of us who are accustomed to read, not by the word, but by the phrase or the line or the paragraph, tend to forget these physical limitations. To take one or two trivial but typical mathematical instances, at what age or intelligence level can we safely begin to use with our pupils summation and product signs:

$$\Sigma\, a_n x^n \text{ instead of } a_0 + a_1 x + a_2 x^2 + \dots$$

and

$$\Pi\, u_n \text{ instead of } u_1 u_2 u_3 \dots ?$$

How much sophistication must be acquired before the solidus is comprehended without a voluntary effort? Incidentally, the answer to this question would be more easily given if printers and mathematicians would co-operate in agreeing on conventions for the use of the solidus and on standard spacing which would allow the formula $x/a + y/b = 1$ to appear in solid text without any fear of ambiguity.

I would not claim that this field of inquiry is of fundamental importance to mathematics, but I would claim that it is of great importance to the teacher of mathematics ; it is a field which has been so far too little explored, probably because it appears humdrum and unspectacular. But there are many of us, well aware that we have not been dowered with the supreme gift of creative mathematical power, who could nevertheless contribute fruitfully by such investigations to the general well-being of our subject.

Royal Naval College, Greenwich

INFORMATION ON MATHEMATICAL EDUCATION IN POLAND

By EDWARD MARCZEWSKI

I do not intend to present here in detail the system of mathematical education in Poland. I do not even intend to consider the question whether such a system exists or not, and why. I intend only, in a ten minutes' communication, to give some information on mathematical education in Poland.

The scheme of education in Poland is as follows :

General School	Fundamental classes :	7 years
	Lyceum classes :	4 years
University :		5 years
Aspiranture :		3 years

The curriculum of the general schools contains elementary mathematics. It does not contain an introduction to calculus, as our experience has been against it. The coordinate method and its applications are included in the programme, but no systematic course on analytic geometry is included.

We have had to face in Polish schools several difficulties but we have had also some success. The number and quality of teachers is not yet adequate, the number of schools increases rapidly and the number of children increases still more rapidly. Some schools, especially in the country, are not sufficiently equipped. And so a fruitful movement is being developed in our schools : the construction of mathematical models and other aids by teachers and pupils for their own schools, and (by exchange) even for others.

This lecture was given at the South Asian Conference on Mathematical Education held on 22-28 February 1956 at the Tata Institute of Fundamental Research, Bombay.

Every year, mathematical competitions, so called "olympic competitions", are organized for pupils of 10th and 11th classes by the Polish Mathematical Society under the sponsorship of the Ministry of Education. The tradition of such competitions is older in the Soviet Union and in Hungary. In Poland their scope is very wide and includes all schools having lyceum classes. In each university centre there exists a special regional committee besides a central committee in Warsaw. The competitions are conducted at three levels: 1. Local competitions—exercises to be solved at home. 2. Competitions at regional level. 3. Central competitions. Each year, the central committee publishes all problems set during the competitions. (I have presented to the Tata Institute a little book containing the material of our fifth olympic competition.)

A serious effort is made by Polish mathematicians for popularizing mathematics, especially among the pupils of the last lyceum class. Each regional section of the Polish Mathematical Society organizes open lectures. These lectures are published in various forms, for instance, as articles in magazines or as separate papers. The new, much enlarged edition of the well-known book of Steinhaus, *Mathematical Snapshots* (under the Polish title *Kalejdoskope Matematyczny*) appeared last year.

The journal "*Matematyka*" for teachers of mathematics and for able pupils is also a result of co-operation between school teachers and university professors. For instance, in its large and interesting section on problems the initials W. S. and H. St. can be found very often: they are Professors Sierpinski and Steinhaus who take part systematically in this enterprise.

Let us consider now universities. Our university programme of mathematics is still under discussion. The programme of the first year is the only one which has remained unchanged, since its establishment a long time ago: it contains calculus, analytical geometry, higher algebra and experimental physics.

Apropos the following courses, it is necessary to emphasize that the character of mathematical universities in Poland after the

2nd World War has been changed essentially to suit the needs of our country and our socialistic economy. First, the liberal system of studies, (which was established after the model of Austrian universities before the 1st World War) was replaced by systematic and compulsory studies, as in Soviet or American universities. And then in order to forge a strong link between mathematics, science in general and applications, various courses on higher analysis and applied mathematics have been introduced. These include also an introduction to set theory, topology, real functions, etc.

A great effort was made and is still being made to give to our university students modern textbooks in Polish on fundamental branches of mathematics. This aim has been almost achieved already.

There are some fundamental traditions of the Polish School of mathematics, which are being closely preserved and carefully developed in our universities : the union of teaching and research, the early beginning of scientific research by gifted students (and the opportunity for such a beginning), and the method of collective research. A great number of seminars common to advanced students, post-graduate fellows and scientific workers is the essential means for the realization of these aims. For instance, there are now in Wroclaw more than 20 seminars organized by the University or by the Mathematical Institute of the Polish Academy of Science. All these seminars are open to all mathematicians, including the students. We strongly think that any barrier between universities and research institutes should be avoided.

In all our educational work, we have had to face several difficulties and problems which have not been mentioned in this communication, because they are common to other countries, and have been discussed in preceding lectures. We have also had some success and the most important one is the emergence of a set of gifted and promising young mathematicians. We hope that they will essentially contribute to the development of our country and to the development of mathematics.

University of Wroclaw

A BRIEF ACCOUNT OF THE PRESENT SITUATION OF MATHEMATICAL EDUCATION IN CHINESE UNIVERSITIES

By H. F. TUAN

In this short speech, I would like to talk exclusively about the situation of Chinese mathematical education at the university level. However, I would like to say at least a few words about the education in elementary schools of six years (beginning from the age of seven) and secondary schools of another six years. Our Government and our people are determined to wipe out completely illiteracy from China in twelve years. By literacy we include also the acquaintance with elementary arithmetic. This is a tremendous task, but we have confidence in carrying it through.

Ever since the founding of the Chinese People's Republic in 1949, especially since 1952, we have been carrying out, on a large scale, what is known as reforms in education.

In 1952, we re-organized our universities and colleges. At present, we have altogether thirteen universities with students in mathematics, of which only Peking University has its department of mathematics and mechanics, with two sub-divisions, mathematics and mechanics. Students in Peking University and a few other universities are to study for five years, while students in the rest will study for four years.

The number of students in mathematics has been greatly increased since 1952. Take for example Peking University. The number of such students in 1952 was only around 100 (being the total number of students in the three mathematical departments which were later incorporated into a single one), while now it is around 400, and next year it will be around 500. The number of post-graduate

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students is not yet large, but it will be greatly increased in the not distant future.

The standard of new students has been greatly elevated since 1953. For example, in my department, among around 200 sophomore students, there are around 15 or 20 who are not satisfied with just doing the work of four regular courses of instruction for the first term, i.e. mathematical analysis, higher algebra, theoretical mechanics, differential equations. They seek to direct their excessive energies to read more advanced books (e.g. Natanson's *Theory of Functions of Real Variables*, Riesz-Nagy's *Lectures on Functional Analysis*) or to do research problems, perhaps, not too elementary (e.g. certain functional equations, infinite products of matrices).

In order to encourage high school students gifted in mathematics to study mathematics after their graduation, we have just started in Peking, what has been done in Hungary, Poland, and the Soviet Union for many years, the Olympic in Mathematics for high school students. Two days before I left Peking, I gave a lecture to around 1,000 of them on " symmetry ". A few weeks earlier, Professor Hua Loo-keng lectured on " Yang Hwei Triangles " (they are usually known to the Western World as " Pascal Triangles ", but Yang Hwei made the discovery much earlier in 1261).

Since 1952, it has been made quite clear that the aim of mathematical education at the university level is to train future research workers and teachers in secondary schools or higher institutions. The Teachers' Colleges, more in number, are to train exclusively teachers for secondary schools. For Peking University and a few other universities, after graduation, most of the students in mathematics will undertake research work or teach in higher institutions.

Since 1952, we have made several curricula for university education in mathematics and now we have had a more stable one. We are greatly benefited from the experience of other countries, especially from the Soviet Union. Of course, we have to make our own arrangements in order to suit our own needs.

I would not go into details about courses of instruction. Just for example, I shall mention (theory of) ordinary differential equations, differential equations of mathematical physics, calculus of variations, and integral equations. For freshmen, we have three courses in mathematics, namely, mathematical analysis, analytic geometry and higher algebra. In some universities there will also be given a course on the history of mathematics, which will not only deal with the development of mathematics in the Western World, but will also put due emphasis on the role played by the ancient cultures of Egypt, India and China.

Through the courses in the first three years, we want to lay a solid foundation to the various important branches of mathematics and also neighbouring sciences like physics and mechanics. In this basis, the students are to get more training in a special branch, e.g. the theory of numbers, functional analysis, or else, through special courses and special seminars. Special courses and seminars to be given will differ from university to university, and will be mainly determined by the needs of our national construction, the development of mathematics, and the research work of professors and teachers.

Students will be required to write small papers in the third and fourth years, and to write a thesis in the fifth year. Of course, here again, the level of the thesis will differ from university to university, and even in the same university from student to student. However, we are going to raise the standard from year to year, so that in the not distant future a thesis to be acceptable must contain something new done by the student. In other words, the students are not just copying from textbooks or taking down what the teachers have said to them.

From what I said above, it is clear that we want to have a high standard for mathematical education in the universities. In fact, from the very aim of the university training—to train research workers and teachers, it is clear that teachers and especially professors themselves have to be research workers. We have well-known

mathematicians, but their numbers are far from enough, and the problem facing teachers and especially professors in the universities is to intensify their researches in mathematics.

We set for ourselves a high standard to strive for. Those universities with better staffs and better students will reach the goal very soon. Others may have some difficulties and may take a longer time in catching up. But all have to come up. We have a huge task, but we shall accomplish it.

Peking University

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

BOMBAY, 22-28 FEBRUARY 1956

RESOLUTION 1

1. The Conference, having taken into account the existing practices in South Asia, dealt with the problems of mathematical education at three levels, undergraduate, graduate and post-graduate, defined as follows: undergraduate education covers primary and secondary school education as well as the Intermediate stage, irrespective of whether some or all of the Intermediate instruction is given at college; graduate education covers the Bachelor's degree, the Master's degree, and their variants; post-graduate education covers all education beyond the graduate stage; and resolved to adopt the proposals detailed under the headings below.

2. PURPOSES. The purpose of teaching at the undergraduate level should be utilitarian. The instruction should be related closely and continuously to the needs, the capacity, and the interests of the pupil.

The purpose of teaching at the level of the Bachelor's degree should be to meet the requirements of society in general, and to provide training for teachers of mathematics in secondary schools.

The purpose of teaching at the level of the Master's degree should be to provide training for professional work in the mathematical sciences, including the work of teaching students for the first degree.

The purpose of teaching at the post-graduate level should be to train students for research, and to fit them for the teaching of advanced mathematics, at least up to the second degree.

3. SUBJECTS OF STUDY. By the end of the primary course, the child should understand simple ideas about numbers and spatial relations.

At the secondary stage, an integrated and simplified course should be taught comprising arithmetic, algebra, and geometry, with the possible addition of simple and essential statistical notions. It is contemplated that special instruction should be offered to those who require it.

Intermediate instruction should include an introduction to analytical geometry, calculus and trigonometry.

Required subjects of study for the Bachelor's degree should be analytical geometry, calculus, algebra, and mechanics. Optional subjects might include numerical methods, principles of statistics, elements of mathematical logic, and higher geometry.

Required subjects of study for the Master's degree should be: real and complex analysis; modern algebra; differential geometry of curves and surfaces; elements of mathematical statistics; methods of mathematical physics; mechanics of continuous media. A variety of optional subjects might be provided in accordance with local conditions.

The subjects of study at the post-graduate stage should include the following : real function theory, including Lebesgue integration, measure theory, probability ; complex function theory including Riemann's mapping theorem ; modern algebra through Galois theory ; theory of topological spaces leading to the study of compact Hausdorff spaces, including Tychonoff's theorem, Urysohn's lemma, and Tietze's extension theorem ; affine and projective geometry in connexion with algebra ; differential geometry.

The formal course-work at this stage should not exceed two years.

It is not contemplated that this course should culminate in a formal examination.

4. Teachers. Teachers of mathematics in primary schools should have a knowledge of mathematics equivalent to that required for the school-leaving examination, and some special training in teaching.

Teachers of mathematics in secondary schools should have a degree, with mathematics as a principal subject, and some special training in teaching.

An appropriate proportion of inspectors of schools, both primary and secondary, should have had experience in the teaching of mathematics.

Teachers of mathematics at the level of the first degree should hold a higher degree with mathematics as a principal subject.

Teachers of mathematics at the level of the second degree should have pursued a course of post-graduate study in mathematics.

Teachers of mathematics at the post-graduate level should be mathematicians with significant experience in research and an adequate background of mathematical knowledge. Teaching duties at the graduate and post-graduate levels should be light enough for the efficient discharge of the teacher's primary obligation to pursue advanced study and research. There is a definite gain in teachers at the post-graduate level participating in graduate instruction.

5. Examinations. The Conference considers that the nature of the examination system has such a powerful influence on the work of the student, and on the character of the teaching, that special attention must be given to its design. The system should give a proper shape and direction to the flow of students through the entire range of the curriculum.

A detailed study of this problem should take into account the following points. Injustice should not be done to the student by compelling him to stake his career on the results of a single examination. The timing of the examinations should be closely related to the different stages of the curriculum. Written examinations should be supplemented by oral examinations, and an evaluation of the student's total performance. At the graduate level, teachers should participate in the examination of their own students.

6. Research contracts. A system of research contracts should be instituted by means of which mathematicians can be supported for limited periods of advanced study or research. The award of such contracts should be based on the scientific recommendations made by qualified referees.

7. Summer schools and seminars. With the purpose of improving the quality of teaching and research in the field of mathematics, encouragement should be given to the organization of a limited number of summer schools and seminars adapted to the needs of teachers at all levels, students at the post-graduate level, and research workers.

8. Scholarships. Scholarships at the under-graduate and graduate levels should be considered in the context of financial support for students in general. An important aspect of this problem, which deserves special attention, is the need to encourage able students to advance to the post-graduate stage. Post-graduate students should receive generous financial assistance either for advanced study or for research.

9. Text-books. The preparation of suitable text-books is an urgent problem confronting the whole of South-Asia. Governments can help in solving it by creating and financing regional text-book committees. These committees should be composed of mathematicians, and empowered to seek out and induce competent authors to write such books. Publication and prescription of text-books should be the function of independent agencies, separate from the regional text-book committees.

RESOLUTION 2

The Conference resolved to set up a Committee for Mathematics in South Asia under the chairmanship of the President, Professor K. Chandrasekharan, with the following members : the Secretary of the International Mathematical Union, an expert mathematician to be nominated jointly by the Chairman and the Secretary of the I.M.U., and representatives of the Governments of Burma, Ceylon, India, Indonesia, Malaya-Singapore, Pakistan, and Thailand. The Conference hereby authorizes its President to take all necessary steps for the prompt constitution of the Committee.

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

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APPENDIX

MATHEMATICAL EDUCATION IN SCHOOLS

Report of a Committee consisting of Professor T. A. A. Broadbent (Chairman), Professor K. R. Gunjikar (co-Chairman), Professor Aung Hla, Mr. S. D. Manerikar, Dr. R. Naidu, Mr. Poerwadi Poerwadisastro, Mr. Rabil Sitasuwana and Miss H. K. Wong.

SUGGESTED CONTENT OF PRIMARY COURSE

General aim : Primary teaching is to be related not to the ideas of the teacher but to the needs and experience of the child.

Topics :

i. Numbers—counting, measuring; number combinations and simple operations. (The child might be encouraged to construct his own multiplication tables and use them till the number combinations become thoroughly familiar). Normally (i) will be covered in the first three years.

ii. Simple ideas about fractions and decimal notations.

iii. Elementary geometrical notions. The child should have familiarity with simple geometrical objects and patterns—tiles, simple models,—giving him *informal* ideas about congruence and symmetry. (No *formal* geometry is to be included).

These topics (ii) & (iii) should be started not later than the 4th year of the primary course.

iv. The use of symbols as abbreviations should be developed during the last year of the course.

v. Pictorial representation of numerical data should be introduced at convenient stages.

Mathematics at the transition stage (one year immediately following the primary stage)

1. Familiarity with arithmetical computation including the use of decimals and fractions must be firmly established early in the first year of the secondary stage, along with their simplest applications related to the experiences of the children.

2. Algebra is to be introduced as generalized arithmetic.

3. Informal geometrical notions acquired in the primary stage should be gathered into a more precise pattern preparatory to formal geometry later in this stage. Graphic representation of familiar data, preferably collected by the child, should be introduced.

Mathematics at the later secondary stage (for all pupils)

This general course should be so designed that the special course intended for pupils with a special aptitude for mathematics should grow out of this course.

1. *Arithmetic*. Development of the operations already studied—introduction of the idea of proportion—applications to civic arithmetic—use of algebraic language to express these ideas.

2. *Statistics*. Introduction to statistics should be made through the collection of appropriate data by the pupils themselves. Ideas of statistics should be carried far enough to enable the pupil to examine these and to appreciate the significance of the results.

3. *Algebra*. Algebra must gradually cease to be generalized arithmetic and become a discipline in its own right, a language which shall enable the pupil to solve more general problems in other fields of mathematical studies. The three laws of algebra and the four rules of operations must be thoroughly appreciated, and their application to elementary problems should be fully worked out.

4. *Geometry*. Geometry should begin to crystallise round certain key concepts and theorems. Pupils should be made to appreciate the fundamental logical pattern in geometry. The notion of similarity

should be introduced and linked up with the notion of proportion in arithmetic. Formal geometry must be subordinated to practical needs which will be met at this point by applications to measurement of areas, volumes, and general ideas of mensuration.

SPECIAL COURSE IN MATHEMATICS (9TH, 10TH AND 11TH YEARS).

Besides the general course already mentioned, this should include the following :

1. *Algebra* : Up to and including the binomial theorem for a positive integral index.

2. *Geometry* : Plane geometry should be made a more systematic study and the logical connection between groups of key concepts and theorems should now be made more explicit. The idea of co-ordinates should be introduced and made to serve as a link between geometry and trigonometry. An introduction to solid geometry should be made.

3. *Calculus* : Calculus should not necessarily be attempted in this course, but an approach to the calculus through simple graphical notions or through simple kinematical ideas is desirable. Whether the approach to the calculus is made graphically or kinematically, the link with kinematical ideas should be made at an early stage.

4. *Statistics* : To be worked up to a stage where the idea of standard deviation has been thoroughly grasped and illustrated by examples.

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30. Miss H. K. Wong

SOUTH ASIAN CONFERENCE ON MATHEMATICAL EDUCATION

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LIST OF PARTICIPANTS

1.	Professor Y. Akizuki	(Japan)
2.	Professor A. D. Alexandrov	(U.S.S.R.)
3.	Professor B. B. Bagi	(India)
4.	Mr. K. Balagangadharan	(India)
5.	Professor Ram Behari	(India)
6.	Dr. H. J. Bhabha	(India)
7.	Professor A. L. Blakers	(Australia)
8.	Professor S. Bochner	(U.S.A.)
9.	Professor E. Bompiani	(Italy)
10.	Professor T. A. A. Broadbent	(U.K.)
11.	Professor N. G. de Bruijn	(The Netherlands)
12.	Professor K. Chandrasekharan	(India)
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60.	Professor N. M. Shah	(India)

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71.	Professor H. F. TUAN	(China)
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73.	Professor R. VAIDYANATHASWAMY	(India)
74.	Professor R. K. VAISH	(India)
75.	Miss H. K. WONG	(Singapore)

ON SEQUENCES WHICH CONTAIN NO REPETITIONS

By DAVID HAWKINS and WALTER E. MIENTKA

THE following problem, which arises out of the consideration of a possible stop-rule in chess, has been solved by Marston Morse and G. A. Hedlund [1]. It is required to construct an unending sequence of the set (a, b, c) containing no repetitions. A repetition is defined as a block of consecutive letters EE composed of two identical sub-blocks E, E consisting of 1, 2, 3, or more letters. We give a different solution from that of Morse and Hedlund.

Let A, B, and C be the following three blocks, of lengths respectively 15, 18 and 16 :

$$\left.\begin{aligned} A &= bacbcacbabcbaca \\ B &= bacbabcbacbcacbaca \\ C &= bacbcabacabcbaca \end{aligned}\right\}. \tag{1}$$

Let T be the transformation $\begin{bmatrix} a \to A \\ b \to B \\ b \to C \end{bmatrix}$, by which the individual letters in a block are replaced by the *blocks* A, B, C. Let $T(U)$ be the block obtained by this transformation from the block U. By iteration we define $T^r(U)$, so that

$$T^r(U) = T\,[T^{r-1}(U)] = T^{r-1}\,[T(U)]. \tag{2}$$

Now, since b is the first letter of B, the block $T^r(B)$ begins with the block $T^{r-1}(B)$. Thus the successive blocks

$$B,\ T(B),\ T^2(B), \ldots, \tag{3}$$

are also successively longer partial sequences of an infinite sequence S, such that

$$T(S) = S. \tag{4}$$

THEOREM. *The infinite sequence S contains no repetitions.*

Let U, V, W, X be variables whose values are A, B or C. Let U^- and U^+ be sub-blocks of the same block U, with initial or terminal letters deleted, so that $NU^- = U$, or $U^+N = U$, N representing the deleted block. In particular we admit the null block O, such that $NU^- = UU^- = UO = U$ and $U^+N = U^+U = OU = U$. Let Y, Z be variables whose values are any *blocks* of the blocks A, B, C.

The proof depends upon showing that a repetition occurring in $T^r(B)$ implies a repetition in $T^{r-1}(B)$.

1. Any sufficiently long block of letters in S is of the form U^-YV^+ and this representation is unique. That is, if a block $E = U^-YV^+ = W^-ZX^+$, then $U^- = W^-$, $V^+ = X^+$, and $Y = Z$. For, S is a sequence of blocks A, B, C, and can be articulated into such blocks in only one way. The contrary case would imply that, for some U, $U = V^-W^+$. But this is false. For each of the blocks A, B, C begins with *bacb*, and this block occurs otherwise only in the central part of B. But the block $B^- =$ *bacbcacbaca* is not identical with any V^+.

2. In a repetition where E is sufficiently long $EE = U^-YV^+U^-YV^+$ and, by the previous remark, V^+U^-, lying between two blocks Y, must itself be A, B, C, or else O. If it is O, then its components $V^+ = O$ and $U^- = O$, and $EE = YY$. If it is A, B or C, then EE is either preceded by V^+ or followed by U^-, and either $V^+U^-YV^+U^-Y$ or $YV^+U^-YV^+U^-$ is of the form ZZ. For let the central block $V^+U^- = W$, while the initial block $NU^- = \overset{*}{U} \neq W$ and the final block $V^+M = \overset{*}{V} \neq W$. Then this implies that $W = \overset{*}{V}{}^+\overset{*}{U}{}^-$. But no one of the blocks A, B, C can be formed by taking an initial block of one, followed by a final block of the other. For no two blocks agree initially in more than six letters, or finally in more than seven, whereas the shortest of them is of length fifteen.

3. We have shown that any block of the form EE in S implies the existence of a block YY, provided E is of sufficient length to

contain a block of the form Y. Now any block of the form YY in S also appears in $T^r(B)$, for some r sufficiently large. But by Remark 1, any block of the form YY in $T^r(B)$ must also be of the form $T(EE)$, where EE is a double block in $T^{r-1}(B)$. Hence a double block in $T^r(B)$ implies the existence of a double block in $T^{r-1}(B)$. In this way we descend to a block EE which is not sufficiently long to contain a complete block Y.

4. The remaining EE may be at longest of the form $U^- VW^+ = U^- W^+ U^- W^+$. If V is null then EE is null and can be ignored. If V is not null then EE is preceded or followed by W^+ or U^-, respectively, by a proof identical with that in Remark 2, and there is a block YY.

5. Hence the remaining EE may be at longest of the form UV and it can be shown by complete enumeration that UV contains no repetitions. This enumeration is rather lengthy and is therefore omitted. With this step the proof is completed.

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University of Colorado
and
University of Massachusetts

RESEARCH PROBLEM NUMBER 22

By M. K. FORT, Jr.

IN the RESEARCH PROBLEMS department of the *Bulletin of the American Mathematical Society* (vol. 60, p. 501), Richard Bellman has proposed the following research problem:

Solve the functional equation

$$f(x) = \max\left(g(x) + f(ax), h(x) + f(bx)\right),$$

given that $0 < a, b < 1$; $h(x)$, $g(x) > 0$; $h(0) = g(0) = 0$; $h'(x)$, $g'(x) > 0$; $h''(x)$, $g''(x) > 0$.

In this paper we obtain a solution f which is continuous, which is defined for all non-negative x, and which satisfies $f(0) = 0$. Such a solution f will be called *admissible*.

We first show that the given equation has at most one admissible solution. To do this, we make use of the following easily proved inequality:

$$|\max(u_1, v_1) - \max(u_2, v_2)| \leqslant \max(|u_1 - u_2|, |v_1 - v_2|)$$

for all real numbers u_1, u_2, v_1, v_2. If f_1 and f_2 are admissible solutions of the given equation, the above inequality yields

$$|f_1(x) - f_2(x)| \leqslant \max(|f_1(ax) - f_2(ax)|, |f_1(bx) - f_2(bx)|).$$

We may now use induction to prove that for each positive integer n,

$$|f_1(x) - f_2(x)| \leqslant |f_1(x_n) - f_2(x_n)|,$$

where x_n is a number of the form $a^p b^q x$, $p + q = n$. Since $x_n \to 0$ as $n \to \infty$, f_1 and f_2 are continuous, and $f_1(0) = f_2(0) = 0$, we obtain $|f_1(x) - f_2(x)| = 0$. Hence $f_1 = f_2$.

It follows from the mean value theorem that $g(a^n x) \leqslant a^n x g'(x)$. Thus the series $\sum_{n=0}^{\infty} g(a^n x)$ converges uniformly on every bounded interval of non-negative real numbers. We define $\phi(x) = \sum_{n=0}^{\infty} g(a^n x)$.

It is easily seen that ϕ is differentiable and has a continuous derivative. Moreover, $\phi(x) = g(x) + \phi(ax)$ for all $x \geqslant 0$. The function ϕ is an auxiliary function which we use in the construction of an admissible solution of the given equation.

We consider three cases.

CASE 1. $g'(0)/(1-a) > h'(0)/(1-b)$.

Let us define $s(x) = g(x) + \phi(ax) - h(x) - \phi(bx)$. If we use the fact that $\phi(x) = g(x) + \phi(ax)$, we see that $\phi'(0) = g'(0)/(1-a)$, and it is easy to show that $s'(0) = ((1-b)\, g'(0)/(1-a)) - h'(0) > 0$. s' is continuous, and consequently there exists $\epsilon > 0$ such that $s'(x) > 0$ for $0 \leqslant x \leqslant \epsilon$. It follows that $s(x) \geqslant 0$ and hence $\phi(x) = \max\,(g(x) + \phi(ax), h(x) + \phi(bx))$ for $0 \leqslant x \leqslant \epsilon$. We define $f(x) = \phi(x)$ for $0 \leqslant x \leqslant \epsilon$. Let $c = \max\,(a, b)$. Now suppose that n is a non-negative integer and $f(t)$ has been defined for $0 \leqslant t \leqslant \epsilon/c^n$ so that

$$f(t) = \max\,(g(t) + f(at), h(t) + f(bt)),$$

we then define $f(x)$ for $0 \leqslant x \leqslant \epsilon/c^{n+1}$ by letting

$$f(x) = \max\,(g(x) + f(ax), h(x) + f(bx)).$$

The right side of the above equation is meaningful since ax and bx are in the interval from 0 to ϵ/c^n. By using this recursive scheme, we define $f(x)$ for all $x \geqslant 0$ and obtain an admissible solution of the given equation.

CASE 2. $g'(0)/(1-a) < h'(0)/(1-b)$.

Since the equation is symmetric in g, a and h, b we obtain an admissible solution as in Case 1.

CASE 3. $g'(0)/(1-a) = h'(0)/(1-b)$.

If λ is any number for which $0 < \lambda < 1$, we can use the results of Case 1 to obtain a function f_λ such that

$$f_\lambda(x) = \max\,(g(x) + f_\lambda(ax), \lambda h(x) + f_\lambda(bx)).$$

We will next prove that $\lim\limits_{\lambda \to 1} f_\lambda(x)$ exists and that the convergence is uniform on every bounded interval of non-negative numbers.

Suppose that $0 < \lambda < 1$ and $0 < \mu < 1$. If we recall the construction used in Case 1, we see that there exists $\delta > 0$ such that $f_\lambda(x) = f_\mu(x) = \phi(x)$ for $0 \leqslant x \leqslant \delta$. We let $c = \max(a, b)$. If we again make use of the inequality

$$|\max(u_1, v_1) - \max(u_2, v_2)| \leqslant \max(|u_1 - u_2|, |v_1 - v_2|),$$

it is easy to prove by induction that

$$|f_\lambda(x) - f_\mu(x)| \leqslant |\lambda - \mu| (h(x) + h(cx) + \dots + h(c^{n-1}x))$$

for $0 \leqslant x \leqslant \delta/c^n$. We define $\psi(x) = \sum_{n=0}^{\infty} h(c^n x)$ for all $x \geqslant 0$. ψ is a continuous function, and

$$|f_\lambda(x) - f_\mu(x)| \leqslant |\lambda - \mu| \psi(x)$$

for $x \geqslant 0$. It now follows from the Cauchy criterion for convergence that $\lim_{\lambda \to 1} f_\lambda(x)$ exists and that the convergence is uniform on every bounded interval. The function f obtained as the limit of f_λ as $\lambda \to 1$ is easily seen to be an admissible solution of the given equation.

The University of Georgia

ON AN ANALOGOUS FOURIER SERIES AND ITS CONJUGATE SERIES

By A. M. CHAK

1. It is well known that if

$$\bar{a}_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt\, dt\,; \quad \bar{b}_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt\, dt\,;$$

and

$$f(t \pm 2\pi) = f(t),$$

then the series $\frac{1}{2}\,\bar{a}_0 + \sum\limits_{n=1}^{\infty} (\bar{a}_n \cos nx - \bar{b}_n \sin nx)$ and its conjugate series $\sum\limits_{n=1}^{\infty} (\bar{a}_n \sin nx - \bar{b}_n \cos nx)$ which, on substituting the above values for $\bar{a}_n$ and $\bar{b}_n$ respectively, become

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(t)\cos n(t-x)\, dt$$

and

$$-\frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(t)\sin n(t-x)\, dt$$

converge to $f(x)$ and $\bar{f}(x)$,

$$\bar{f}(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{\psi(t)}{2\,\mathrm{tg}\; t/2}\, dt,$$

respectively under certain conditions.

Mitra [2] and Chak [1], by the help of Bessel functions, defined the *analogous Fourier series* and the *analogous conjugate Fourier series* respectively. They showed that the conditions of ordinary convergence and of summability $(C, 1)$ of the series $\frac{1}{2}\, p_0(x) + \sum\limits_{n=1}^{\infty} p_n(x)$ to $f(x)$ and of its conjugate series $\sum\limits_{n=1}^{\infty} q_n(x)$ to $\bar{f}(x)$,

$$\bar{f}(x) = -\int_0^{\pi} \frac{\psi^*(t)}{2\,\mathrm{tg}\,\frac{1}{2}t}\;\frac{dt}{\sqrt{(\pi - t^2)}},$$

where $\psi^*(t)$ is got from $\psi(t) = f(x+t) - f(x-t)$ by substituting t for $\pi \sin(\frac{1}{2} t)$, are the same as the respective well-known conditions for the Fourier series and its conjugate series, where

$$\begin{matrix} p_n \\ q_n \end{matrix}(x) = \pm \tfrac{1}{2} \int_0^{2\pi} f(t) \begin{matrix} \cos \\ \sin \end{matrix} \Big[n\pi \sin \tfrac{1}{2}(t-x) \Big] dt.$$

The object of the present paper is to define another analogous Fourier series and its conjugate series and to show that if we have two sequences defined by

$$\begin{matrix} a_n \\ b_n \end{matrix}(x) = \pm \tfrac{1}{4} \int_0^{\pi} \Big[f(x+t) \pm f(x-t) \Big] \begin{matrix} \cos \\ \sin \end{matrix} \Big[n\pi \tan \tfrac{1}{4} t \Big] dt, \quad (1)$$

then the two series

$$\mathfrak{S}[f] \equiv \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n(x)$$

and

$$\bar{\mathfrak{S}}[f] \equiv \sum_{n=1}^{\infty} b_n(x)$$

converge respectively to $f(x)$ and to $\bar{f}(x)$,

$$\bar{f}(x) = -\pi \int_0^{\pi} \frac{\psi^*(t)}{2 \operatorname{tg} \frac{1}{2} t} \frac{dt}{\pi^2 + t^2},$$

where $\psi^*(t)$ is got from $\psi(t)$ by the substitution of t for $\pi \tan(\frac{1}{4} t)$ under the same respective conditions as the Fourier series and its allied series; further, that the respective conditions for the analogous Fouries series and its allied series to be summable $(C, 1)$ to $f(x)$ and $\bar{f}(x)$ respectively are also the same in both the cases.

It is interesting to note here that (i) $\bar{f}(x)$ is not exactly the same for the conjugate Fourier series and the two analogous conjugate Fourier series, i.e. the one defined here and the other as defined by the author [1].

The author is attempting to prove more general results for a much larger class of analogous Fourier series.

2. Convergence of the analogous Fourier series and its conjugate.

It is easy to see [4, Ch. II] that the nth partial sums $S_n(x)$ and $\bar{S}_n(x)$ respectively of the series $\mathfrak{S}[f]$ and $\bar{\mathfrak{S}}[f]$ can be written as

$$S_n(x) = \tfrac{1}{4} \int_0^\pi [f(x+t) + f(x-t)]\, A_n(t)\, dt$$

and

$$\bar{S}_n(x) = -\tfrac{1}{4} \int_0^\pi [f(x+t) - f(x-t)]\, \bar{A}_n(t)\, dt, \tag{2}$$

where A_n and $\bar{A}_n$, the *analogous Dirichlet's kernel* and the *analogous Dirichlet's conjugate kernel* respectively, are given by

$$\left.\begin{aligned} A_n(t) &= \frac{\sin[(n+\frac{1}{2})\pi \tan(\frac{1}{4}t)]}{2\sin[\frac{1}{2}\pi\tan(\frac{1}{4}t)]} \\ &\text{and} \\ \bar{A}_n(t) &= \frac{\cos[\frac{1}{2}\pi\tan(\frac{1}{4}t)] - \cos[(n+\frac{1}{2})\pi\tan(\frac{1}{4}t)]}{2\sin[\frac{1}{2}\pi\tan(\frac{1}{4}t)]}. \end{aligned}\right\} \tag{3}$$

Let

$$S_n^*(x) = S_n(x) - \tfrac{1}{2} a_n;\ \bar{S}_n^*(x) = \bar{S}_n(x) - \tfrac{1}{2} b_n; \tag{4}$$

and we have

$$\left.\begin{aligned} S_n^*(x) &= \tfrac{1}{4}\int_0^\pi [f(x+t)+f(x-t)]\, A_n^*(t)\, dt \\ &\text{and} \\ \bar{S}_n^*(x) &= -\tfrac{1}{4}\int_0^\pi [f(x+t)-f(x-t)]\, \bar{A}_n^*(t)\, dt, \end{aligned}\right\} \tag{5}$$

where

$$\left.\begin{aligned} A_n^*(t) &= \frac{\sin[n\pi\tan(\frac{1}{4}t)]}{2\,\mathrm{tg}[\frac{1}{2}\pi\tan(\frac{1}{4}t)]} \\ &\text{and} \\ \bar{A}_n^*(t) &= \frac{1-\cos[n\pi\tan(\frac{1}{4}t)}{2\,\mathrm{tg}[\frac{1}{2}\pi\tan(\frac{1}{4}t)]}. \end{aligned}\right\} \tag{6}$$

If $f \equiv 1$, then

$$\sigma_n^* = \sigma_n - \int_0^{\pi/4} \cos\,[\,n\,\pi \tan t\,]\,dt. \tag{7}$$

where

$$\sigma_n = \tfrac{1}{4} \int_0^{\pi} \frac{\sin\,[(n+\frac{1}{2})\,\pi \tan(\frac{1}{4}\,t)]}{\sin\,[\frac{1}{2}\,\pi \tan(\frac{1}{4}\,t)]}\,dt$$

$$= \frac{\pi}{4} + 2 \sum_{k=1}^{n} \int_0^{\pi/4} \cos\,[k\,\pi \tan t]\,dt, \tag{8}$$

σ_n^* and σ_n being the values of S_n^* and S_n respectively for $f \equiv 1$.

Let us now find the Fourier series for the function which is equal to $1/(1+x^2)$ in the interval $(-1, 1)$. We have ([3], p. 633)

$$\frac{1}{1+x^2} = \frac{\pi}{4} + 2 \sum_{n=1}^{\infty} \cos n\pi x \int_0^{\pi/4} \cos\,[k\,\pi \tan t]\,dt. \tag{9}$$

For $x = 0$, this gives

$$1 = \frac{\pi}{4} + 2 \sum_{n=1}^{\infty} \int_0^{\pi/4} \cos\,[n\pi \tan t]\,dt. \tag{10}$$

From (8) and (10) we see that $\sigma_n \to 1$ as $n \to \infty$, and hence (7) shows that $\sigma_n^* \to 1$, as $n \to \infty$.

Putting $I_n^* = S_n^*(x) - \sigma_n^* f(x)$ and $I_n = \bar{S}_n^*$, we have

$$\left.\begin{aligned} I_n^* &= \tfrac{1}{4} \int_0^{\pi} \frac{\sin\,[n\pi \tan(\frac{1}{4}\,t)]}{2\,\mathrm{tg}[\frac{1}{2}\,\pi \tan(\frac{1}{4}\,t)]}\,\phi(t)\,dt \\ &\text{and} \\ \bar{I}_n^* &= -\tfrac{1}{4} \int_0^{\pi} \frac{\{1-\cos\,[n\pi \tan(\frac{1}{4}\,t)]\}}{2\,\mathrm{tg}\,[\frac{1}{2}\,\pi \tan(\frac{1}{4}\,t)]}\,\psi(t)\,dt, \end{aligned}\right\} \tag{11}$$

where

$$\phi(t) \equiv f(x+t) + f(x-t) - 2f(x)$$

and

$$\psi(t) \equiv f(x+t) - f(x-t).$$

Putting $\pi \tan(\frac{1}{4}\,t) = u$ in (11), we have

$$\left.\begin{aligned} I_n^* &= \pi \int_0^\pi \frac{\phi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \frac{\sin nt}{\pi^2+t^2}\,dt \\ \text{and}\quad \bar{I}_n^* &= -\pi \int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \frac{1-\cos nt}{\pi^2+t^2}\,dt, \end{aligned}\right\} \tag{12}$$

where

$$\phi^*(u) \equiv \phi(t) \text{ and } \psi^*(u) \equiv \psi(t).$$

Let

$$\left.\begin{aligned} J_n^* &= \frac{1}{\pi} \int_0^\pi \frac{\sin nt}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)}\,\phi^*(t)\,dt \\ \text{and}\quad \bar{J}_n^* &= -\frac{1}{\pi} \int_0^\pi \frac{1-\cos nt}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)}\,\psi^*(t)\,dt, \end{aligned}\right\} \tag{13}$$

then

$$\left.\begin{aligned} I_n^* - J_n^* &= \pi \int_0^\pi \frac{\sin nt}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)}\,\phi^*(t) \left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right] dt \\ \text{and}\quad \bar{I}_n^* - \bar{J}_n^* &= -\pi \int_0^\pi \frac{1-\cos nt}{2\mathrm{tg}(\tfrac{1}{2}t)}\,\psi^*(t) \left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right] dt. \end{aligned}\right\} \tag{14}$$

Since $\dfrac{\phi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \left[\dfrac{1}{\pi^2+t^2} - \dfrac{1}{\pi^2}\right]$ and $\dfrac{\psi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \left[\dfrac{1}{\pi^2+t^2} - \dfrac{1}{\pi^2}\right]$

are summable in $(0, \pi)$, it follows by the Riemann-Lebesgue theorem that as $n \to \infty$, $I_n^* - J_n^* \to 0$, and

$$\bar{I}_n - \bar{J}_n^* \to -\pi \int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right] dt,$$

on the sole assumption that $f(x)$ is summable in $(0, \pi)$. Hence, as $n \to \infty$,

$$\left.\begin{aligned} &S_n^* \to f(x), \text{ if } J_n^* \to 0, \\ \text{and}\quad &\bar{S}_n^* \to -\pi \int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)} \frac{dt}{\pi^2+t^2}, \text{ if } \bar{J}_n^* = -\frac{1}{\pi} \int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\tfrac{1}{2}t)}\,dt. \end{aligned}\right\} \tag{15}$$

Now J_n^* and $\bar{J}_n^*$ differ respectively from the partial sums S_n^* and $\bar{S}_n^*$ of the Fourier series of $f(x)$ and its allied series only in that $\phi^*(t)$ replaces $f(x+t)+f(x-t)-2f(x) \equiv \phi(t)$ and $\psi^*(t)$ replaces $f(x+t)-f(x-t) \equiv \psi(t)$, and it is obvious that from the relation $u = \pi \tan(\frac{1}{2}t)$ all the usual conditions for S_n^* and $\bar{S}_n^*$ will be applicable to J_n^* and $\bar{J}_n^*$ respectively, and vice-versa. Hence we have the *analogous Dini's test : If the first of the integrals*

$$\int_0^\pi \frac{|\phi^*(t)|}{2\,\mathrm{tg}\,(\frac{1}{2}t)}\,dt, \quad \int_0^\pi \frac{|\psi^*(t)|}{2\,\mathrm{tg}\,(\frac{1}{2}t)}\,dt, \tag{16}$$

is finite, then $\mathfrak{S}[f]$ converges at the point x to the sum $f(x)$. If the second integral is finite then $\bar{\mathfrak{S}}[f]$ converges at the point x to the value which we shall denote by $\bar{f}(x)$, where

$$\bar{f}(x) = -\pi \int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\frac{1}{2}t)}\,\frac{dt}{\pi^2+t^2}.$$

For the proof we notice that $S_n^* - \sigma_n^* f(x)$ and $\bar{S}_n^* - f(x)$, by equation (3), are respectively the Fourier sine and cosine coefficients of integrable functions.

3. **Summability-$(C,1)$ of the analogous Fourier series.** Let $\lambda_n(x) = \lambda_n(x; f)$ be the first arithmetic means of $\{S_n(x)\}$. Using formulae (2) and (3) we see that

$$\begin{aligned}\lambda_n(x) &= (S_0+S_1+\dots+S_{n-1})/n \\ &= \tfrac{1}{4}\int_0^\pi [f(x+t)+f(x-t)]C_n(t)\,dt,\end{aligned} \tag{17}$$

where

$$C_n(t) = (A_0+A_1+\dots+A_{n-1})/n. \tag{18}$$

Taking $f(x) \equiv 1$, we have

$$\lambda_n(x) - \bar{\lambda}_n f(x) = \tfrac{1}{4}\int_0^\pi \phi(t)\,C_n(t)\,dt, \tag{19}$$

where

$$\bar{\lambda}_n = (\sigma_0+\sigma_1+\dots+\sigma_{n-1})/n. \tag{20}$$

Therefore

$$\tilde{\lambda}_n(x) = \frac{1}{n}\left\{\frac{1}{4}\pi n + 2\sum_{k=1}^{n-1}(n-k)\int_0^{\pi/4}\cos\,[k\pi\tan t]\,dt\right\}$$

$$= \tfrac{1}{4}\pi + 2\sum_{k=1}^{n-1}\int_0^{\pi/4}\cos\,[k\pi\tan t]\,dt -$$

$$-\frac{2}{n}\sum_{k=1}^{n-1}k\int_0^{\pi/4}\cos\,[k\pi\tan t]\,dt. \tag{21}$$

As $n \to \infty$, the second limit tends to zero and hence from (10), $\lambda_n(x) \to 1$. Therefore

$$I_n \equiv \lambda_n - \tilde{\lambda}_n f(x) = \frac{1}{8n}\int_0^{\pi}\frac{\sin^2\,[\frac{1}{2}\,n\pi\tan\frac{1}{4}\,t]}{\sin^2\,[\frac{1}{2}\,\pi\tan(\frac{1}{4}\,t)]}\,\phi(t)\,dt. \tag{22}$$

Let $\pi\tan(\frac{1}{4}\,t) = u$. Then we define

$$J_n = \frac{1}{\pi n}\int_0^{\pi}\frac{\sin^2(\frac{1}{2}\,nt)}{2\sin^2(\frac{1}{2}\,t)}\,\phi^*(t)\,dt. \tag{23}$$

Therefore

$$I_n - J_n = \frac{\pi}{n}\int_0^{\pi}\frac{\sin^2(\frac{1}{2}\,nt)}{2\sin^2(\frac{1}{2}\,t)}\,\phi^*(t)\left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right]dt. \tag{24}$$

Now

$$|I_n - J_n| \leqslant \frac{\pi}{n}\int_0^{\pi}\frac{\phi^*(t)}{2\sin^2(\frac{1}{2}\,t)}\left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right]dt, \tag{25}$$

and the integral on the right exists. Therefore $I_n - J_n \to 0$ as $n \to \infty$, and the condition for $(C, 1)$-summability becomes

$$\lim_{n\to\infty}\frac{1}{n}\int_0^{\pi}\frac{\sin^2(\frac{1}{2}\,nt)}{\sin^2(\frac{1}{2}\,t)}\,\phi^*(t)\,dt = 0, \tag{26}$$

where, as before, $\phi^*(t)$ replaces $\phi(t)$. Hence all the usual theorems on $(C, 1)$-summability of Fourier series follow immediately.

We will now state the following important theorem, sketching the outline of its proof.

Analogous Fejer's theorem. *If the limits $f(x \pm 0)$ exist, $\mathfrak{S}[f]$ is summable $(C, 1)$ at the point x to the value $\frac{1}{2}\,[f(x+0) + f(x-0)]$.*

In particular, if f is continuous at x, $\mathfrak{S}[f]$ is summable there to the value $f(x)$; if f is continuous at every point of an interval $I = \langle a, b\rangle$, $\mathfrak{S}[f]$ is uniformly summable in I.

Multiplying the denominator of $A_k(t)$ in (3) by $2 \sin [\frac{1}{2} \pi \tan (\frac{1}{4} f)]$ and replacing the products of sines by differences of cosines, we find that

$$n\, C_n(t) = \sum_{k=0}^{n-1} \frac{\sin [(k + \frac{1}{2}) \pi \tan (\frac{1}{4} t)]}{2 \sin [\frac{1}{2} \pi \tan (\frac{1}{4} t)]}$$

$$= \frac{1}{2} \left(\frac{\sin [\frac{1}{2} n \pi \tan (\frac{1}{4} t)]}{\sin [\frac{1}{2} \pi \tan (\frac{1}{4} t)]} \right)^2 \tag{27}$$

The expression $C_n(t)$, called the *analogous Fejer's kernel*, is a positive kernel, since it satisfies the following three properties :

(i) $C_n(t) > 0$,

(ii) $\frac{1}{2} \int_0^\pi C_n(t)\, dt = 1$,

(iii) $M_n(\delta) \to 0$ as $n \to \infty$ for every $\delta > 0$,

where $M_n(\delta) \equiv \max |C_n(t)| \equiv \max C_n(t)$ for $\delta \leqslant t \leqslant \pi$, $n = 0, 1, 2, 3, \ldots$

Basing our proof of the theorem exactly on the lines of Zygmund ([4], Ch. III) and noticing that $C_n(t)$ is a positive kernel we can easily prove the analogous Fejer's theorem. I think it is useless to repeat the arguments of Zygmund ([4], Ch. III,) here.

From the analogous Fejer's theorem all the usual corollaries which are true in the case of Fourier series, can easily be derived for the analogous Fourier series.

4. Summability-$(C, 1)$ of the analogous conjugate Fourier series.

Let

$$\rho_n = (\bar{S}_0 + \bar{S}_1 + \ldots + \bar{S}_{n-1})/n, \tag{28}$$

then

$$\rho_n = -\frac{1}{8n}\int_0^\pi \frac{n\cos[\frac{1}{2}n\tan(\frac{1}{4}t)] - \sum_{k=1}^{n}\cos[\frac{1}{2}(2k-1)\pi\tan(\frac{1}{4}t)]}{\sin[\frac{1}{2}\pi\tan(\frac{1}{4}t)]} \times$$

$$\times\, \psi(t)\, dt$$

$$= -\frac{1}{8n}\int_0^\pi \frac{n\sin[\pi\tan(\frac{1}{4}t)] - \sin[n\pi\tan(\frac{1}{4}t)]}{2\sin^2[\frac{1}{2}\pi\tan(\frac{1}{4}t)]}\,\psi(t)\,dt; \qquad (29)$$

that is,

$$\rho_n + \tfrac{1}{8}\int_0^\pi \frac{\psi(t)\,dt}{\mathrm{tg}\,[\frac{1}{2}\pi\tan(\frac{1}{4}t)]} = \frac{1}{8n}\int_0^\pi \frac{\sin[n\pi\tan(\frac{1}{4}t)]}{2\sin^2[\frac{1}{2}\pi\tan\frac{1}{4}t]}\,\psi(t)\,dt. \qquad (30)$$

Let $\pi\tan(\frac{1}{4}t) = u$, and evaluating as in § 2, we have

$$\bar{I}_n \equiv \rho_n + \frac{\pi}{2}\int_0^\pi \frac{\psi^*(t)}{\mathrm{tg}\,(\frac{1}{2}t)}\,\frac{dt}{\pi^2+t^2} = \frac{\pi}{2n}\int_0^\pi \frac{\sin nt}{2\sin^2(\frac{1}{2}t)}\cdot\frac{\psi^*(t)\,dt}{\pi^2+t^2}. \qquad (31)$$

Putting

$$\bar{J}_n = \frac{1}{2\pi n}\int_0^\pi \frac{2\sin^2(\frac{1}{2}t)}{\sin nt}\,\psi^*(t)\,dt, \qquad (32)$$

we have

$$\bar{I}_n - \bar{J}_n = \frac{\pi}{2n}\int_0^\pi \frac{\sin nt}{2\sin^2(\frac{1}{2}t)}\,\psi^*(t)\left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right]dt. \qquad (33)$$

Therefore

$$\left|\bar{I}_n - \bar{J}_n\right| \leqslant \frac{\pi}{2n}\int_0^\pi \frac{1}{2\sin^2(\frac{1}{2}t)}\,\psi^*(t)\left[\frac{1}{\pi^2+t^2} - \frac{1}{\pi^2}\right]dt, \qquad (34)$$

and as the integral on the right exists we have $\bar{I}_n - \bar{J}_n \to 0$, as $n \to \infty$. Hence the condition for $\bar{\mathfrak{S}}\,[f]$ to be summable $(C, 1)$ to the sum

$$\bar{f}(x) = -\pi\int_0^\pi \frac{\psi^*(t)}{2\,\mathrm{tg}\,(\frac{1}{2}t)}\,\frac{dt}{\pi^2+t^2}$$

becomes

$$\lim_{n\to\infty}\frac{1}{n}\int_0^\pi \frac{\sin nt}{\sin^2(\frac{1}{2}t)}\,\psi^*(t)\,dt \to 0, \qquad (35)$$

where, as before, $\psi^*(t)$ replaces $\psi(t)$.

Thus all the usual theorems on $(C, 1)$-summability of conjugate Fourier series follow easily for the analogous conjugate Fourier series.

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Lucknow University

A NOTE ON ENTIRE FUNCTIONS DEFINED BY DIRICHLET SERIES

By QAZI IBADUR RAHMAN

1. CONSIDER the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n\, e^{s\lambda_n},\ \lambda_{n+1} > \lambda_n,\ \lambda_1 \geqslant 0,\ \lim_{n\to 1} \lambda_n = \infty,\ s = \sigma + it.$$

It defines in its half-plane of convergence, a holomorphic function. Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$.

Let $\mu(\sigma)$ be the maximum of $|a_n|\, e^{\sigma\lambda_n}\,(n = 1, 2, \ldots)$, and $M(\sigma)$, $M'(\sigma)$, $M''(\sigma), \ldots$ the l. u. b.'s of $|f(s)|$, $|f'(s)|$, $|f''(s)|, \ldots$ where σ is a constant inferior to σ_a. Let $\lambda_{N(\sigma)}$ be the λ_n corresponding to the maximum term for $\mathscr{R}(s) = \sigma$. $\lambda_{N(\sigma)}$ is evidently a nondecreasing function of σ.

It is well known that $\log M(\sigma)$ is a convex function of σ and therefore $\dfrac{M(\sigma - k)}{M(\sigma)}$ will decrease steadily with σ for every $k > 0$.

We prove

THEOREM 1. *For an integral function* $f(s) = \sum\limits_{n=1}^{\infty} a_n\, e^{s\lambda_n}$ *of lower order* λ $(0 \leqslant \lambda \leqslant \infty)$,

$$\lambda = \liminf_{n\to\infty} \frac{\log\log \mu(\sigma)}{\sigma} = \liminf_{n\to\infty} \frac{\log \lambda_{N(\sigma)}}{\sigma}$$

provided that

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0.$$

THEOREM 2. *If* $a_n \geqslant 0$,

$$\frac{M(\sigma)}{M(\xi)} > \frac{M'(\sigma)}{M'(\xi)} > \frac{M''(\sigma)}{M''(\xi)} > \cdots > \frac{M^n(\sigma)}{M^n(\xi)}$$

for every $\xi > \sigma$.

LEMMA. *Let*

(i) $\phi(x)$ *be a positive increasing function;*

(ii) $\liminf_{x\to\infty} \dfrac{\log \phi(x)}{x} = \alpha \quad (0 \leqslant \alpha \leqslant \infty).$

Then corresponding to each pair of positive numbers β, γ *satisfying the inequalities*

(iii) $\alpha < \beta, \alpha/\beta < \gamma < 1,$

there is a sequence $x_1, x_2, \ldots$ *tending to infinity such that*

(iv) $\phi(x) < e^{\beta x} \qquad (\gamma x_n \leqslant x \leqslant x_n).$

For let $x_1, x_2, \ldots$ be a sequence such that

$$\frac{\log \phi(x_n)}{x_n} < \beta \gamma.$$

Then, if $\gamma x_n \leqslant x \leqslant x_n$,

$$\log \phi(x) \leqslant \log \phi(x_n) < \beta \gamma x_n \leqslant \beta x \text{ or } \phi(x) < e^{\beta x}.$$

2. PROOF OF THEOREM 1. Let

$$\liminf_{n\to\infty} \frac{\log \lambda_{N(\sigma)}}{\sigma} = \alpha. \tag{1}$$

Also, we have [1, p. 68], for $\sigma < \sigma_a$,

$$a_n e^{\sigma\lambda_n} = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{T} e^{-it\lambda_n} f(\sigma + it)\, dt,$$

and therefore

$$|a_n| e^{\sigma\lambda_n} \leqslant M(\sigma),$$

and consequently

$$\mu(\sigma) \leqslant M(\sigma). \tag{2}$$

And [1, p. 67]

$$\log \mu(\sigma) = \frac{\lambda_1 G_2 - \lambda_2 G_1}{\lambda_2 - \lambda_1} + \int_{(G_2 - G_1)/(\lambda_2 - \lambda_1)}^{\sigma} \lambda_{N(\sigma)}\, d\sigma,$$

$$\log \mu(\sigma) = \log \mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \lambda_{N(\sigma)}\, d\sigma \tag{3}$$

give

$$\lambda_{N(\sigma)}\,\sigma \leqslant \int_{\sigma}^{2\sigma} \lambda_{N(\sigma)}\,d\sigma < \log \mu(2\,\sigma) \leqslant \log M(2\,\sigma),$$

whence

$$\frac{\log \lambda_{N(\sigma)}}{\sigma} + \frac{\log \sigma}{\sigma} < \frac{\log\log M(2\,\sigma)}{\sigma},$$

$$\alpha \leqslant \lambda \qquad (0 \leqslant \lambda \leqslant \infty). \tag{4}$$

Now suppose that $\alpha < \infty$. $\lambda_{N(\sigma)}$ is an increasing function and so by the Lemma, if $\alpha < \beta$, $\frac{\alpha}{\beta} < \gamma < 1$, there is a sequence $\sigma_1, \sigma_2, \ldots$ for which

$$\frac{\log \lambda_{N(\sigma)}}{\sigma} < \beta \qquad (\gamma\,\sigma_n \leqslant \sigma \leqslant \sigma_n). \tag{5}$$

Take positive numbers δ, ϵ such that $\gamma < \delta < 1$, $\gamma/\delta < \epsilon < 1$ and write $s_n = \delta\,\sigma_n$, so that

$$\gamma\,\sigma_n = (\gamma/\delta)\,s_n < \epsilon\,s_n < s_n.$$

By (3),

$$\log \mu(s_n) = \log \mu(\epsilon\,s_n) + \int_{\epsilon s_n}^{s_n} \lambda_{N(\sigma)}\,d\sigma$$

and

$$\log \mu(\epsilon\,s_n) < \epsilon\,\lambda_{N(\epsilon s_n)}\,s_n,$$

so that

$$\begin{aligned}\log \mu(s_n) &\geqslant \log \mu(\epsilon\,s_n) + \lambda_{N(\epsilon s_n)} \int_{\epsilon s_n}^{s_n} d\sigma \\ &= \log \mu(\epsilon\,s_n) + (1-\epsilon)\,s_n\,\lambda_{N(\epsilon s_n)} \\ &> \log \mu(\epsilon\,s_n) + \frac{1-\epsilon}{\epsilon}\log \mu(\epsilon\,s_n) \\ &= \frac{1}{\epsilon}\log \mu(\epsilon\,s_n),\end{aligned}$$

$$-\epsilon\log \mu(s_n) < -\log \mu(\epsilon\,s_n),$$

$$\log \mu(s_n) - \epsilon \log \mu(s_n) < \log \mu(s_n) - \log \mu(\epsilon\, s_n) = \int_{\epsilon s_n}^{s_n} \lambda_{N(\sigma)}\, d\sigma.$$

Thus using (5),

$$(1-\epsilon) \log \mu(s_n) < \int_{\epsilon s_n}^{s_n} e^{\beta\sigma}\, d\sigma = \frac{e^{\beta s_n} - e^{\beta \epsilon s_n}}{\beta}$$

$$= e^{\beta s_n} \frac{1 - e^{-(1-\epsilon)\beta s_n}}{\beta}. \tag{6}$$

Also we find [1, p. 73], if

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0,$$

that

$$\frac{\log \mu(s_n)}{\log M(s_n)} \to 1 \quad \text{as} \quad s_n \to \infty. \tag{7}$$

Form (6), (7) we deduce that

$$\liminf_{n\to\infty} \frac{\log\log M(s_n)}{s_n} = \liminf_{n\to\infty} \frac{\log\log \mu(s_n)}{s_n} \leqslant \beta,$$

whence

$$\lambda \leqslant \alpha. \tag{8}$$

(4) and (8) are equivalent to

$$\liminf_{n\to\infty} \frac{\log \lambda_{N(\sigma)}}{\sigma} = \lambda \qquad (0 \leqslant \lambda \leqslant \infty).$$

Proof of Theorem 2. Since $a_n \geqslant 0$,

$$f(\sigma) = M(\sigma), \quad f'(\sigma) = M'(\sigma), \quad \text{and so on.}$$

Hence for $\sigma > -\infty$,

$$\frac{d}{d\sigma}\left\{\frac{M(\sigma-k)}{M(\sigma)}\right\} = \frac{M'(\sigma-k)\, M(\sigma) - M'(\sigma)\, M(\sigma-k)}{\{M(\sigma)\}^2} < 0.$$

So

$$\frac{M'(\sigma-k)}{M'(\sigma)} < \frac{M(\sigma-k)}{M(\sigma)}.$$

Writing σ for $\sigma - k$ and ξ for σ we get

$$\frac{M(\sigma)}{M(\xi)} > \frac{M'(\sigma)}{M'(\xi)}, \qquad (\sigma < \xi).$$

Obviously $\dfrac{M'(\sigma)}{M'(\xi)}$ is also a decreasing function of σ and applying the above argument to this case we get

$$\frac{M'(\sigma)}{M'(\xi)} > \frac{M''(\sigma)}{M''(\xi)}.$$

Hence

$$\frac{M(\sigma)}{M(\xi)} > \frac{M'(\sigma)}{M'(\xi)} > \frac{M''(\sigma)}{M''(\xi)} > \dots > \frac{M^n(\sigma)}{M^n(\xi)}.$$

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Muslim University
Aligarh

A NOTE ON THE UNION CURVATURE OF THE CURVES OF A RIEMANNIAN SPACE

By MILEVA PRVANOVITCH

1. Introduction. A curve of a subspace V_n imbedded in a Riemannian space V_m having the property that, at every point, its osculating geodesic surface contains the tangent vector to a curve of a congruence of curves through this point, is called a *union curve* of the subspace V_n relative to the given congruence. In the special case, when the congruence is normal to the subspace V_n, union curves are geodesic curves of the subspace V_n. Union curves of a hypersurface V_n imbedded in a Riemannian space V_{n+1} were defined and studied by Springer [3], whilst Mishra [2] considered the union curves of a subspace immersed in a Riemannian space V_m.

The object of this paper is the examination of the union curvature of any curve C of subspace V_n imbedded in a Riemannian space V_m. It will be demonstrated that the expression for the union curvature of the curve C, in the special case, has the form given by Springer [3] and Mishra [4].

2. Necessary relations. Let V_n be a Riemannian space with coordinates x^i and metric $g_{ij}\,dx^i\,dx^j$, immersed in a Riemannian space V_m with coordinates y^α and metric $a_{\alpha\beta}\,dy^\alpha\,dy^\beta$. (Unless stated otherwise, Latin indices take the values $1, 2, \ldots, n$; early Greek indices (α, β, γ) take the values $1, 2, \ldots, m$, whilst later Greek indices (σ, ν, τ) the values $n+1, \ldots, m$). If both metrics are taken to be positive definite,

$$g_{ij} = a_{\alpha\beta}\, y^\alpha_{,i}\, y^\beta_{,j},$$

where a comma followed by an index indicates covariant differentiation with respect to the g's.

A set of $m-n$ unit, mutually orthogonal, vectors $N^\alpha_{\nu|}$ which, at a point of the subspace V_n, are normal to V_n, satisfy the relations

$$a_{\alpha\beta}\, N^{\alpha}_{\nu|}\, N^{\beta}_{\sigma|} = \delta^{\nu}_{\sigma},$$

$$a_{\alpha\beta}\, y_{,i}\, N^{\beta}_{\nu|} = 0.$$

Consider a system of $m-n$ congruences of curves in V_m, one curve of each congruence passing through every point of the subspace V_n. Let $\lambda^{\alpha}_{\tau|}$ be the contravariant components of a unit vector in the direction of a curve of the congruence $\lambda_{\tau|}$, the index τ fixes the congruence in the given system. In general, this vector is not normal to V_n, and therefore it can be decomposed into a component tangential to V_n and a component normal to V_n, i.e. it may be written

$$\lambda^{\alpha}_{\tau|} = t^{i}_{\tau|}\, y^{\alpha}_{,i} + \sum_{\nu} l_{\nu\tau}\, N^{\alpha}_{\nu|}. \tag{1}$$

Taking into consideration the fact that $\lambda^{\alpha}_{\tau|}$ is a unit vector, it can be demonstrated that

$$1 - g_{ij}\, t^{i}_{\tau|}\, t^{j}_{\tau|} = \sum l^{2}_{\nu\tau|}. \tag{2}$$

The differential equations of a union curve of the subspace V_n relative to the congruence $\lambda_{\tau|}$ are given by [2] in the form

$$l_{\nu\tau|}\, p^{i} = \Omega_{\nu|mn} \frac{dx^m}{ds} \frac{dx^n}{ds} \left(t^{i}_{\tau|} - t_{\tau|l} \frac{dx^l}{ds} \frac{dx^i}{ds} \right),$$

where $\dfrac{dx^i}{ds}$ are the components of the unit tangent vector to the curve, p^i are the components of the curvature vector of the curve relative to V_n, and $\Omega_{\nu|mn}$ are the components of the tensor of the second fundamental form of the subspace V_n.

For a curve C of the subspace V_n the vector with contravariant components

$$\eta^{i} = p^{i} - \frac{\Omega_{\nu|mn} \dfrac{dx^m}{ds} \dfrac{dx^n}{ds}}{l_{\nu\tau|}} \left(t^{i}_{\tau|} - t_{\tau|l} \frac{dx^l}{ds} \frac{dx^i}{ds} \right) \tag{3}$$

is the union curvature vector of the curve C relative to the congruence $\lambda_{\tau|}$. The magnitude k_u of this vector is the union curvature of the curve C.

3. Union curvature of the curve. From (3) it follows that

$$k_u^2 \equiv g_{ij}\,\eta^i\,\eta^j = g_{ij}\left[p^i - \Omega_{\nu|mn}\,\frac{dx^m}{ds}\,\frac{dx^n}{ds}\,(l^{\nu\tau 1})^{-1}\left(t^i_{\tau|} - t_{\tau|l}\,\frac{dx^l}{ds}\,\frac{dx^i}{ds}\right)\right]\times$$

$$\times\left[p^j - \Omega_{\nu|mn}\,\frac{dx^m}{ds}\,\frac{dx^n}{ds}\,(l^{\nu\tau|})^{-1}\left(t^j_{\tau|} - t_{\tau|k}\,\frac{dx^k}{ds}\,\frac{dx^j}{ds}\right)\right],$$

so that

$$k_u^2 = k_g^2 - 2\,\Omega_{\nu|mn}\,\frac{dx^m}{ds}\,\frac{dx^n}{ds}\,(l_{\nu\tau|})^{-1}\,g_{ij}\,p^i\,t^j_{\tau|} +$$

$$+\left(\Omega_{\nu|mn}\,\frac{dx^m}{ds}\,\frac{dx^n}{ds}\,(l_{\nu\tau|})^{-1}\right)^2\left(g_{ij}\,t^i_{\tau|}\,t^j_{\tau|} - t_{\tau|l}\,t_{\tau|k}\,\frac{dx^l}{ds}\,\frac{dx^k}{ds}\right),$$

since

$$g_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds} = 1 \text{ and } g_{ij}\,\frac{dx^i}{ds}\,p^j = 0,$$

whilst k_g is the magnitude of the vector p^i.

Let us denote by $\alpha_{\tau|}$ the angle between the vector $t^i_{\tau|}$ and the tangent vector to the curve, and by $\beta_{\tau|}$ the angle between the vector $t^i_{\tau|}$ and the vector p^i. Then with respect to (2), we have

$$g_{ij}\,t^i_{\tau|}\,\frac{dx^j}{ds} = \left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}}\cos\alpha_{\tau|}, \tag{4}$$

$$g_{ij}\,t^i_{\tau|}\,p^j = k_g\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}}\cos\beta_{\tau|}. \tag{5}$$

Hence, for the union curvature of the curve C relative to the congruence $\lambda_{\tau|}$, we get

$$k_u^2 = k_g^2 - 2\,\frac{\Omega_{\nu|mn}\,\dfrac{dx^m}{ds}\,\dfrac{dx^n}{ds}}{l^{\nu\tau|}}\,k_g\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}}\cos\beta_{\tau|} +$$

$$+\left(\frac{\Omega_{\nu|mn}\,\dfrac{dx^m}{ds}\,\dfrac{dx^n}{ds}}{l_{\nu\tau|}}\right)^2\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}}\sin^2\alpha_{\sigma|}. \tag{6}$$

If the curves of the congruence $\lambda_{\tau|}$ are normal to V_n, the vectors with components $t^i_{\tau|}$ are zero, and the parameters $l_{\nu\tau|}$ satisfy the relation

$$\sum_{\nu} l^2_{\nu\tau|} = 1.$$

Hence, from (6),

$$k_u = k_g$$

that is, the union curvature becomes the geodesic curvature of the curve C.

If the vector $t^i_{\tau|}$ is expressed linearly in terms of the vectors dx^i/ds and p^i, viz. if

$$t^i_{\tau|} = a_{\tau|} \frac{dx^i}{ds} + b_{\tau|} p^i, \tag{7}$$

then

$$g_{ij} t^i_{\tau|} t^j_{\tau|} = a^2_{\tau|} + k^2_g b^2_{\tau|},$$

or

$$1 - \sum_{\nu} l^2_{\nu\tau|} = a^2_{\tau|} + k^2_g b^2_{\tau|}.$$

On the other hand, we get from (7)

$$g_{ij} p^i t^j_{\tau|} = g_{ij} p^i \left(a_{\tau|} \frac{dx^j}{ds} + b_{\tau|} p^j \right) = k^2_g b_{\tau|},$$

so that, from (5)

$$\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}} \cos \beta_{\tau|} = b_{\tau|} k_g.$$

Furthermore, from (7)

$$g_{ij} t^i_{\tau|} \frac{dx^j}{ds} = g_{ij} \frac{dx^j}{ds} \left(a_{\tau|} \frac{dx^i}{ds} + b_{\tau|} p^i \right) = a_{\tau|},$$

so that, from (4)

$$\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)^{\frac{1}{2}} \cos \alpha_{\tau|} = a_{\tau|}.$$

Hence

$$\left(1 - \sum_{\nu} l^2_{\nu\tau|}\right)(\cos^2 \alpha_{\tau|} + \cos^2 \beta_{\tau|}) = 1 - \sum_{\nu} l^2_{\nu\tau|},$$

whence it follows that

$$\cos^2 \alpha_{\tau|} + \cos^2 \beta_{\tau|} = 1, \tag{8}$$

or

$$\cos^2 \beta_{\tau|} = \sin^2 \alpha_{\tau|}. \tag{9}$$

Consequently, when the condition (7) is satisfied, the union curvature (6) of the curve is expressed in the form

$$k_u = k_g - \frac{\Omega_{\nu|mn} \dfrac{dx^m}{ds} \dfrac{dx^n}{ds}}{l_{\nu\tau|}} \left(1 - \sum_{\nu} l_{\nu\tau|}\right)^{\frac{1}{2}} \sin \alpha_{\tau|}. \tag{10}$$

However, this is the expression given by Springer [3] and Mishra [1] for the union curvature of the curve C.

We shall demonstrate that the condition (7) is both necessary and sufficient for the existence of the equation (9), i.e. that the union curvature of the curve C can be expressed in the form (10). For this purpose, let us consider a system of n mutually orthogonal vectors dx^i/ds, p^i, $\xi^i_{r|}$ $(r = 3, \dots, n)$ in V_n. Then we have

$$g_{ij} \frac{dx^i}{ds} \xi^j_{r|} = 0, \quad g_{ij}\, p^i\, \xi^j_{r|} = 0, \quad g_{ij}\, \xi^i_{r|}\, \xi^j_{s|} = \delta^r_s \ (r,\ s = 3, \dots, n).$$

Since the vector $t^i_{\tau|}$ is a vector of the subspace V_n, it can be expressed in the form

$$t^i_{\tau|} = a_{\tau|} \frac{dx^i}{ds} + b_{\tau|}\, p^i + \sum_r c_{\tau r|}\, \xi^i_{r|} \qquad (r = 3, \dots, n).$$

Hence

$$g_{ij}\, t^i_{\tau|}\, t^j_{\tau|} = g_{ij} \left(a_{\tau|} \frac{dx^i}{ds} + b_{\tau|}\, p^i + \sum_r c_{\tau r|}\, \xi^i_{r_1} \right) \times$$

$$\times \left(a_{\tau|} \frac{dx^j}{ds} + b_{\tau|}\, p^j + \sum_r c_{\tau|}\, \xi^j_{r|} \right)$$

from which

$$1 - \sum_{\nu} l^2_{\nu\tau|} = a^2_{\tau|} + k_g^2\, b^2_{\tau|} + \sum_r c^2_{r\tau|}.$$

Further, we have

$$g_{ij}\, p^i\, t^j_{\tau|} = g_{ij}\, p^i \left(a_{\tau|}\, \frac{dx^j}{ds} + b_{\tau|}\, p^j + \sum_r c_{r\tau|}\, \xi^j_{r|} \right) = b_{\tau|}\, k_g^2,$$

and with regard to (5),

$$\left(1 - \sum_{\nu} l^2_{\nu\tau|} \right) \cos \beta_{\tau|} = k_g\, b_{\tau|}.$$

Likewise

$$g_{ij}\, \frac{dx^i}{ds}\, t^j_{\tau|} = g_{ij}\, \frac{dx^i}{ds} \left(a_{\tau|}\, \frac{dx^j}{ds} + b_{\tau|}\, p^j + \sum_r c_{r\tau}\, \xi^j_{r_1} \right) = a_{\tau|},$$

and, by use of (4), we get

$$\left(1 - \sum_{\nu} l^2_{\nu\tau|} \right)^{\frac{1}{2}} \cos \alpha_{\tau|} = a_{\tau|}.$$

Hence

$$\left(1 - \sum l^2_{\nu\tau|} \right) (\cos^2 \alpha_{\tau|} + \cos^2 \beta_{\tau|}) = a^2_{\tau|} + k_g^2\, b^2_{\tau|},$$

so that

$$\left(a^2_{\tau|} + k_g^2\, b^2_{\tau|} + \sum_r C^2_{r\tau|} \right) (\cos^2 \alpha_{\tau|} + \cos^2 \beta_{\tau|}) = a^2_{\tau|} + k_g^2\, b^2_{\tau|}.$$

Therefore, for the existence of the relation (8) we would have

$$C_{r\tau|} = 0 \quad \text{for every } r.$$

Hence, we can say:

The necessary and sufficient condition that the relation (9) *be satisfied, i.e. that the union curvature of a curve can be expressed in the form* (10), *is that the vector* $t^i_{\tau|}$ *belong to the geodesic surface determined by vectors* $\dfrac{dx^i}{ds}$ *and* p^i, *i.e. that*

$$t^j_{\tau|} = a_{\tau|}\, \frac{dx^i}{ds} + b_{\tau|}\, p^i. \tag{7}$$

When the relation (7) *is not satisfied, the union curvature of a curve has the form* (6).

The condition (7) is always satisfied when V_n is a surface, i.e. when $n = 2$. Hence we have the theorem:

The union curvature of a curve of a subspace V_n is expressed in the form (10).

This reveals, by development of the differential equations of the union curves [2], that, for the union curves, the condition (7) is always satisfed. Hence, we can say:

The geodesic curvature of a union curve can be expressed in the form

$$k_g = \frac{\Omega_{\nu|mn} \dfrac{dx^m}{ds} \dfrac{dx^n}{ds}}{l_{\nu\tau|}} \left(1 - \sum l^2_{\nu\tau|}\right)^{\frac{1}{2}} \sin \alpha_{\tau|}\,.$$

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Belgrade

On three intersecting circles

By NATHAN ALTSHILLER COURT, *University of Oklahoma*

1. Introduction. *a.* Given three mutually intersecting circles (A), (B), (C), let (M) be their orthogonal circle, and let

$$P, P'; Q, Q'; R, R' \tag{a}$$

be the pairs of points of intersection of the pairs of circles (B), (C); (C), (A); (A), (B), respectively.

The points of (a) located on each of the given circles are listed in the following

TABLE 1

(A)		Q, Q';	R, R'
(B)	P, P';		R, R'
(C)	P, P';	Q, Q'.	

b. In what follows we shall consider the relations between the triangles and circles determined by the points of intersection (a) of the given circles.

In connection with this study it will be purposeful to consider the triangle $m_a m_b m_c$ formed by the radical axes m_a, m_b, m_c of the pairs of circles (M), (A); (M), (B); (M), (C), respectively. We shall refer to $m_a m_b m_c$ as the *radical triangle* of the circles (A), (B), (C).

The triangle ABC formed by the centers A, B, C of the three given circles shall be said to be the *central triangle* of those circles.

The vertices $A' = m_b m_c$, $B' = m_c m_a$, $C' = m_a m_b$ of the radical triangle lie, respectively, on the radical axes $p = PP'$, $q = QQ'$, $r = RR'$. It is readily seen that the radical triangle and the central triangle are polar reciprocal with respect to the circle (M), and therefore are perspective [L. Cremona; *Projective geometry*, 3rd ed. § 336].

2. Perspective triangles. *a.* Taking one point from each of the three pairs of points of intersection (a) we obtain eight triangles to which we shall refer as the *triangles of intersection*. They are listed in the following

TABLE 2

	I	II	III	IV
Triangles of intersection	$PQR, P'Q'R'$	$PQ'R', P'QR$	$P'QR', PQ'R$	$P'Q'R, PQR'$
Axes of similitude	$s = DEF$	$s_p = DE'F'$	$s_q = EF'D'$	$s_r = FD'E'$
Circles of intersection	$(I), (I')$	$(I_p), (I_p')$	$(I_q), (I_q')$	$(I_r), (I_r')$
Circumcenters	$I, I',$	I_p, I_p'	I_q, I_q'	I_r, I_r'

The eight triangles of intersection are grouped into four pairs I, II, III, IV of *complementary triangles of intersection*. Two such triangles have no vertex in common, and thus involve all six points of intersection.

b. The three radical axes p, q, r (§ 1*b*) meet in the orthogonal center M of the circles (A), (B), (C), hence the four pairs of complementary triangles of intersection I, II, III, IV are perspective. Let s, s_p, s_q, s_r denote their respective axes of perspectivity. If

$$D = (QR,\ Q'R'),\ D' = (QR',\ Q'R);$$
$$E = (RP,\ R'P'),\ E' = (RP',\ R'P);$$
$$F = (PQ,\ P'Q'),\ F' = (PQ',\ P'Q),$$

then we have, by Desargues's theorem,

$$s = DEF,\ s_p = DE'F',\ s_q = EF'D',\ s_r = FD'E'.$$

3. The quadrilateral of intersection. *a.* The four axes of perspectivity s, s_p, s_q, s_r determine a complete quadrilateral (S) to which we shall refer as the *quadrilateral of intersection* of the three circles (A), (B), (C). Its three pairs of opposite vertices are:

$$D = ss_p,\ D' = s_q s_r;\ E = ss_q,\ E' = s_r s_p;\ F = ss_r,\ F' = s_p s_q \ (\S\, 2\text{b}).$$

b. The four points Q, Q', R, R' determine a complete quadrangle inscribed in the circle (A) (see Table 1, § 1*a*) having the points

M, D, D' for vertices of its diagonal triangle (§ 2), hence the points D, D' lie on the polar of M with respect to (A) [Cremona, l. c., § 260]. Now the circles (M), (A) being orthogonal (§ 1), the polar of M for (A) coincides with the radical axis m_a (§ 1) of those two circles, hence the line m_a joins the two opposite vertices D, D' of the complete quadrilateral (S) (§ 3*a*) and is therefore a side of the diagonal triangle of (S).

Similarly the lines m_b, m_c join the points E, E'; F, F', respectively. Thus: *The diagonal triangle of the complete quadrilateral of intersection of three intersecting circles coincides with the radical triangle of those circles.*

c. The two vertices D, D' of the diagonal triangle MDD' of the complete quadrangle $QQ'RR'$ (§ 3*b*) are conjugate with respect to the circle (A) and lie on the radical axis m_a of the two (orthogonal) circles (M), (A), hence D, D' are also conjugate with respect to the circle (M) [N. A. Court : *College geometry*, § 429].

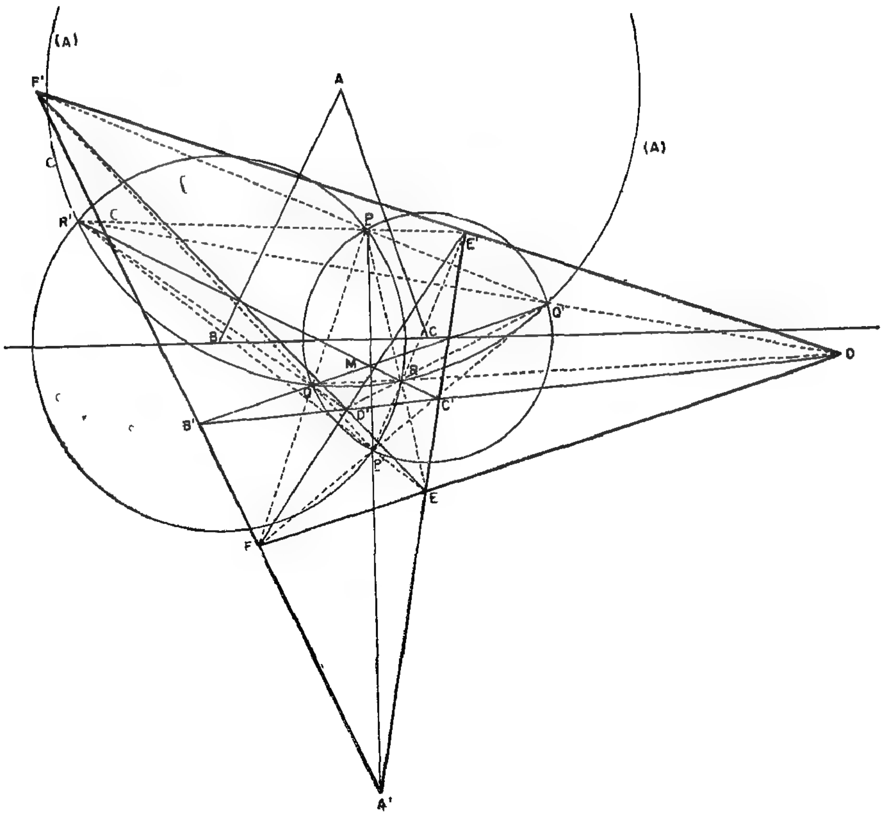
Similarly for the pairs of points E, E'; F, F'. Thus: *The pairs of opposite vertices of the quadrilateral of intersection of three intersecting circles are pairs of conjugate points with respect to the orthogonal circle of the three given circles.*

4. **Circles of intersection.** *a.* The circumcircles of the triangles of intersection (§ 2*a*) will be said to be the *circles of intersection* of the three given circles.

Two circles of intersection will be said to be *complementary*, if their inscribed triangles of intersection are complementary.

b. The points P, P' lie on the circle (B) (see Table 1) and are collinear with the center M of the circle (M) orthogonal to (B), hence P, P' are inverse points with respect to (M). The pairs of points Q, Q'; R, R' are inverse with respect to (M), for analogous reasons.

Thus the circles $(I) = PQR$, $(I') = P'Q'R'$ are inverse with respect to (M), and (M) is a circle of antisimilitude of (I), (I') [*College geometry*, l.c., §§ 543, 544].



Similarly for the other pairs of complementary circles of intersection of the given circles (A), (B), (C). Thus : *The orthogonal circle of three intersecting circles is a circle of antisimilitude of each of the four pairs of complementary circles of intersection of the given circles.*

c. The points P, P' of the circles (I), (I'), inverse with respect to (M), are antihomologous points on those two circles [*College geometry*, l.c., § 526*b*]. Similarly for the pairs of points Q, Q'; R, R'. Hence the pairs of corresponding sides of the two complementary triangles of intersection PQR, $P'Q'R'$ are antihomologous chords of the two circles (I), (I') and therefore interesect on their radical axis [*College geometry*, l.c., § 434]. Thus the radical axis

of (I), (I') coincides with the axis of perspectivity of the triangles PQR, $P'Q'R'$.

Similarly for the other pairs of complementary circles of intersection. Hence : *The sides of the quadrilateral of intersection of three intersecting circles are the radical axes of the four pairs of complementary circles of intersection of the three given circles.*

d. The three circles (M), (I), (I') being coaxal (§ 4*b*), their three centers M, I, I' are collinear. Now M is the center of perspectivity of the two triangles PQR, $P'Q'R'$, and their axis of perspectivity coincides with the radical axis of the two circles (I), (I') (§ 4*c*), hence : *The line joining the circumcenters of the two complementary triangles of intersection of three intersecting circles passes through the center of perspectivity of the two triangles and is perpendicular to their axis of perspectivity.*

e. It follows from the preceding proposition (§ 4*d*), that if u, u_p, u_q, u_r are the perpendiculars from M upon the sides s, s_p, s_q, s_r of the quadrilateral (S) (§ 3*a*), we have the following triads of collinear points (cf. Table 2) :

$$u = MII',\ u_p = MI_p I'_p,\ u_q = MI_q I'_q,\ u_r = MI_r I'_r.$$

5. The centers of similitude of the circles of intersection. *a.* The center M of the circle of antisimilitude (M) of the circles (I), (I') (§ 4*b*) is a center of similitude of those two circles, hence their second center of similitude is the harmonic conjugate M_0 of M with respect to the centers I, I', and the circle (M_0) having M_0 for center and coaxal with (I), (I') is their second circle of antisimilitude. Now the two circles of antisimilitude (M), (M_0) are orthogonal [*College geometry*, l.c., § 483], hence the center M_0 of (M_0) coincides with the pole, for (M), of the radical axis s of the two circles (§ 4*c*).

It may be shown in an analogous fashion that the second centers of similitude, besides the point M, of the pairs of complementary circles of intersection (I_p), (I'_p) ; (I_q), (I'_q) ; (I_r) (I'_r) (cf. Table 2) are the poles M_p, M_q, M_r of their respective radical axes s_p, s_q, s_r

with respect to the circle (M). Thus : *Given three intersecting circles, any two of their complementary circles of intersection have for their two centers of similitude :* (i) *the center of the orthogonal circle of the three given circles, and* (ii) *the pole, for the latter circle, of the radical axis of the two circles of intersection considered.*

b. The complete quadrangle (m) determined by the four points M_0, M_p, M_q, M_r, (§ 5*a*) is the polar reciprocal, for the circle (M), of the quadrilateral (S) (§ 3). Now the diagonal triangle of (S) is the radical triangle $A'\ B'\ C' = m_a\ m_b\ m_c$ of the three given circles (A), (B), (C) (§ 3*b*), hence the diagonal triangle of (m) is the polar reciprocal, for (M), of $A'\ B'\ C'$, that is, the central triangle ABC (§ 1*b*). Thus : *The central triangle of three intersecting circles is the diagonal triangle of the complete quadrangle determined by the four centers of similitude (other than the orthogonal centre of the given cirlces) of the four pairs of complementary circles of intersection of the three given circles.*

c. The side $M_0\,M_p$ of (m) is the polar of the point $D = ss_p$ (§ 3*a*) for the circle (M) and therefore passes through the conjugate D' of D for (M) (§ 3*c*). Similarly for the other sides of (m). Thus : *The complete quadrangle* (m) (§ 5*b*) *is circumscribed about the complete quadrilateral of intersection* (S) *of the given circles.*

d. The polar $M_0\,M$ of the point D for (M) passes through the vertex A of the diagonal triangle ABC of the complete quadrangle (m) (§ 5*b*) and through the point D' (§ 5*c*). Thus the polar, for (M), of the point D, situated on the line s, joins the vertices A, D' of the triangles ABC, $D'E'F'$ and passes through the pole M_0 of the line s for (M). Similarly for the polars $M_0M_qE'B$, $M_0M_rF'C$ of the points E, F of s for (M).

Analogous considerations apply to the other sides of (S). Thus : *Given three intersecting circles, the triangle formed by any three of their four axes of intersection is perspective to their central triangle.*

The center of perspectivity of the two triangles is the pole of the fourth axis of intersection with respect to the orthogonal circles of the given circles.

6. The associated triad of circles. *a.* DEFINITION. The point of intersection A' of the two radical axes m_b, m_c (§1*b*) is the orthogonal centre of the three circles (M), (B), (C). Let (A') denote their orthogonal circle.

We have similarly the orthogonal circles (B'), (C') of the two triads of circles (M), (C), (A); (M), (A), (B).

The circles (A'), (B'), (C') form the associated triad of circles of the three given circles (A), (B), (C).

Observe that from the preceding relations and § 1 it follows that the circles (A), (B), (C), respectively, are the orthogonal circles of the triads of circles (M), (B'), (C'); (M), (C'), (A'); (M), (A'), (B').

b. The two points D, D' being conjugate with respect to (M) (§ 3*c*), the circle (DD') having DD' for diameter is orthogonal to (M) [*College geometry* l.c. § 387]. The center of (DD') obviously lies on the line $DD' = m_a = B'C'$ (§ 3*b*), hence the three circles (B'), (C'), (DD'), with collinear centers and orthogonal to the same circle (M), are coaxal [*College geometry*, l.c., § 458].

Moreover, the two opposite vertices D, D' of the quadrilateral (S) are separated harmonically by the vertices B', C' of the diagonal triangle A' B' C' of (S) (§ 3*b*), hence (DD') is the circle of similitude and D, D' are the centers of similitude of the circles (B'), (C') [*College geometry*, § 481].

Likewise the analogous circles (EE'), (FF') are, respectively, the circles of similitude of the pairs of circles (C'), (A'); (A'), (B'). Thus : *The pairs of opposite vertices of the quadrilateral of intersection of three given intersecting circles are the three pairs of centers of similitude of the triad of associated circles of the given circles taken two by two.*

It follows that *the sides of the quadrilateral* (S) *are the four axes of similitude of the associated triad of circles.*

c. The three circles of similitude (DD'), (EE') (FF') of the three circles (A'), (B'), (C') taken in pairs (§ 6*b*) are orthogonal to

(M) and to the circumcircle of the central traingle $A'B'C'$ of the latter circles (*College geometry*, l.c. § 496). The radical axis of the three circles of similitude is thus determined by the centers of the two circles (M) and $(A'B'C')$.

On the other hand, the line of centers of the three circles of similitude considered is the Newton line of the complete quadrilateral (S), hence : *The perpendicular from the orthogonal center of three intersecting circles upon the Newton line of their quadrilateral of intersection passes through the circumcenter of their radical triangle.*

7. Circles of antisimilitude. *a.* The circles (D), (D') coaxal with the circles (B'), (C') and having for centers the centers of similitude D, D' of the latter circles are the two circles of antisimilitude of (B'), (C').

The analogous pairs of circles (E), (E'); (F), (F') are the circles of antisimilitude of the pairs of circles (C'), (A'); (A'), (B'), respectively.

The two circles (A), (M) are orthogonal to each other (§ 1*b*); and both are orthogonal to (B') and (C') (§ 6*a*), hence they are also orthogonal, to the circles (D), (D') coaxal with (B'), (C'). Now the two circles of antisimilitude (D), (D') are orthogonal to each other, hence the four circles (A), (M), (D), (D') are mutually orthogonal. Consequently the four points A, M, D, D' form an orthocentric group of four triangles, and one of the latter four circles is imaginary. [N. A. Court, *Annals of Math.* 1927, 367-368].

The circle (A) is real, by assumption, hence the two circles (D), (D') are both real, if, and only if, the circle (M) is imaginary. Similarly for the pairs of circles (E), (E') ; (F), (F'). The triad of associated circles are in that case real and intersecting.

If (M) is real, the triad of associated circles are either real and non-intersecting, or they are imaginary. In either case only their three external circles of antisimilitude are real.

b. The circle (D) is orthogonal to (M) (§ 7*a*) and has its center on the radical axis s of the three coaxal circles (I), (I'), (M) (§ 4),

hence (D) is also orthogonal to both (I) and (I'). Furthermore, since the point D also lies on the radical axis s_p of the two complementary circles of intersection (I_p), (I'_p) (Table 2), (D) is also orthogonal to the latter two circles, for similar reasons.

Similarly for the other circle of antisimilitude (D') of the two circles (B'), (C').

Analogous considerations apply to the circles of antisimilitude (E), (E') of the two circles (C'), (A'), and again to the circles of antisimilitude (F), (F') of the circles (A'), (B'). Thus: *Given three intersecting circles, each circle of antisimilitude of the associated triad of circles is orthogonal to four of the circles of intersection of the three given circles.*

c. The six circles of antisimilitude of the three circles (A'), (B'), (C') taken in pairs are orthogonal to $6 \times 4 = 24$ circles of intersection of the circles (A), (B), (C). But the latter circles have only eight circles of intersection, hence each circle of intersection is counted $24/8 = 3$ times.

This result is readily verified and rendered more precise by considering the figure. The circle (I) is orthogonal to (D), and, for analogous reasons, to the circles (E) and (F).

Similarly for the other circles of intersection. Hence: *Given three intersecting circles, each of their eight circles of intersection is orthogonal to three of the circles of antisimilitude of the triad of circles associated to the given circles: namely to those circles of antisimilitude whose centers lie on the radical axis of the circle of intersection considered and its complementary circle of intersection. The latter circle is orthogonal to the same three circles of antisimilitude.*

8. Orthogonal circles. The radical triangle of three intersecting circles coincides with the central triangle of those circles, if, and only if, the three given circles are real and mutually orthogonal.

It follows that three mutually orthogonal real circles coincide with their associated triad of cirles (§ 6*a*), that the diagonal triangle of their quadrilateral of intersection conicides with the

central triangle of the given circles (§ 3*b*), etc. This subject, however, is more readily treated directly [N. A. Court, *American Math. Monthly*, 1955].

Kinematics of a rigid body

By P. S. Rau, *Tirupathi*

1. We shall derive the fundamental formula of Rigid Dynamics, viz. $v = w \times r$ where w is the angular velocity vector of the rigid body moving with one point O of it fixed, and r is the position vector of any point P of the body (referred to O as origin) whose velocity vector at an instant t is v. We shall also derive the celebrated theorem of Coriolis for relative motion, viz. the acceleration vector of a particle moving on a rigid body which is itself moving with one point O of it fixed is the sum of the acceleration vector relative to the rigid body, the 'transport acceleration vector' and the 'Coriolis acceleration vector'. Throughout this paper, we denote the first and second derivatives of a quantity (scalar or vector) by suffixes 1 and 2 respectively.

2. Let the fixed point O of the rigid body be taken as the origin of an orthogonal right-handed system of axes OX, OY, OZ fixed in the body. Then as the rigid body moves relative to the fixed point O, the axes OX, OY, OZ move along with it and the position of the rigid body at any instant t is completely determined by the position of the three moving axes. If P be any point of the rigid body whose coordinates referred to OX, OY, OZ are respectively a, b, c and if i, j, k be unit vectors along the directions OX, OY, OZ respectively, then the position vector of the point P referred to O as origin of vectors is given by

$$r = ai + bj + ck.$$

Since the point P and the axes OX, OY, OZ are fixed relative to the body, the coordinates a, b, c are constant throughout the motion and it is only the unit vectors i, j, k that are changing as the body moves. Thus the velocity vector of the point P at any instant t is given by

$$v = r_1 = ai_1 + bj_1 + ck_1.$$

Differentiating the identical relation $i^2 = 1$ with respect to t, we find $i.i_1 = 0$, which shows that i_1 is in the plane of j and k. Similarly we can see that j_1 is in the plane of k and i and that k_1 is in the plane of i and j. Thus we can write

$$i_1 = \lambda_{ij}\, j + \lambda_{ik}\, k,\; j_1 = \lambda_{jk}\, k + \lambda_{ji}\, i,\; k_1 = \lambda_{ki}\, i + \lambda_{kj}\, j. \qquad \text{(A).}$$

Differentiating the identical relations $i.j = j.k = k.i = 0$ with respect to t, we obtain $i.j_1 + j.i_1 = 0, j.k_1 + k.j_1 = 0, k.i_1 + i.k_1 = 0$, which reduce to

$$\lambda_{ji} + \lambda_{ij} = 0,\; \lambda_{kj} + \lambda_{jk} = 0,\; \lambda_{ik} + \lambda_{ki} = 0, \qquad \text{(B)}$$

in view of the relations (A). From (A) and (B) we can write the velocity vector of the point P in the form

$$\begin{aligned} v = r_1 &= ai_1 + bj_1 + ck_1 \\ &= a(\lambda_{ij}\, j - \lambda_{ki}\, k) + b(\lambda_{jk}\, k - \lambda_{ij}\, i) + c(\lambda_{ki}\, i - \lambda_{jk}\, j) \\ &= (\lambda_{jk}\, i + \lambda_{ki}\, j + \lambda_{ij}\, k) \times (ai + bj + ck) \\ &= w \times r, \end{aligned} \qquad \text{(C)}$$

where $w = \lambda_{jk}\, i + \lambda_{ki}\, j + \lambda_{ij}\, k$.

3. It is evident that the vector $w = \lambda_{jk}\, i + \lambda_{ki}\, j + \lambda_{ij}\, k$ occurring in the formula (C) for the velocity vector v of any point P is independent of the position of the point P for it is constructed from the coefficients λ_{ij}, λ_{jk}, λ_{ki} occurring in the values of i_1, j_1, k_1 only. However, we can see that w is independent of the system of OX, OY, OZ also as below :

Let i', j', k' be unit vectors respectively along the directions OX', OY', OZ' of any other right-handed system of rectangular axes

(through O) fixed in the rigid body. Applying the result (C) successively to $r = i'$, $r = j'$, $r = k'$, we have $i'_1 = w \times i'$, $j'_1 = w \times j'$, $k'_1 = w \times k'$. Also if the coordinates of the point P referred to the new system of axes be a', b', c' respectively, we have $r = a'i' + b'j' + c'k'$.

Hence we get

$$
\begin{aligned}
v = r_1 &= a'i'_1 + b'j'_1 + c'k'_1 \\
&= (a'w \times i') + (b'w \times j') + (c'w \times k') \\
&= w \times (a'i' + b'j' + c'k') \\
&= (w \times r).
\end{aligned}
$$

If $w' = \lambda_{j'k'}\, i' + \lambda_{k'i'}\, j' + \lambda_{i'j'}\, k'$ be constructed from the system i', j', k' in the same way as $w = \lambda_{jk}\, i + \lambda_{ki}\, j + \lambda_{ij}\, k$ is obtained from the system i, j, k, we can write $v = r_1 = w' \times r$ analogous to the formula (C). Thus we have $w \times r = w' \times r$ which gives $(w - w') \times r = 0$. This is true for every r and hence we get $w = w'$.

4. Thus the vector w occurring in the velocity vector of any point of the body is independent of the position of the point as also of the system of axes chosen and is therefore a property of the rigid body as a whole at the instant t. Furthermore, v $(= w \times r)$ is zero if and only if $r = \lambda w$. Therefore at the instant t, the motion of the rigid body is just the same as though the axis, which has the direction of w, were fixed in space. We therefore speak of the line through O having the direction of w as the instantaneous axis of rotation of the rigid body at the instant t, and of the vector w as the angular velocity vector of the rigid body at the instant t.

5. Coriolis Theorem. *Let P be a particle moving on a rigid body which is itself moving with one point O of it fixed. As in §2, O is taken as the origin of a unit orthogonal right-handed system of vectors i, j, k fixed in the body. Then the position vector of P referred to O as origin of vectors is given by*

$$r = ai + bj + ck.$$

As the particle P is moving relative to the rigid body the scalars a, b, c, as also the unit vectors i, j, k change with the time t. And so the velocity vector of the particle P at any instant t is given by

$$v = r_1 = (ai + bj + ck)_1 = (a_1 i + b_1 j + c_1 k) + (ai_1 + bj_1 + ck_1)$$
$$= (a_1 i + b_1 j + c_1 k) + (w \times r), \qquad \text{(D)}$$

in view of (C).

$ai_1 + bj_1 + ck_1$ represents the velocity vector of the point of the rigid body which coincides with the position of the particle at the instant t and is called the 'transport velocity vector' of the particle P at the instant t. $a_1 i + b_1 j + c_1 k$ represents the velocity vector of the particle P relative to the rigid body. Thus the velocity vector of the particle P is the sum of the velocity vector relative to the rigid body and the 'transport velocity vector'.

From (D) we have

$$(a_1 i + b_1 j + c_1 k)_1 = (a_2 i + b_2 j + c_2 k) + w \times (a_1 i + b_1 j + c_1 k).$$

Differentiating the relation (D) with respect to t, we find

$$A = r_2 = (a_2 i + b_2 j + c_2 k) + (w_1 \times r) + w \times (w \times r) +$$
$$+ 2w \times (a_1 i + b_1 j + c_1 k).$$

$(a_2 i + b_2 j + c_2 k)$ represents the acceleration vector of the particle P relative to the rigid body. $(w_1 \times r) + w \times (w \times r)$ represents the acceleration vector of that point of the rigid body which coincides with the position of the particle at the instant t and is called the 'transport acceleration vector' of the particle P at the instant t. The vector $2w \times (a_1 i + b_1 j + c_1 k)$ is termed the 'Coriolis acceleration vector' of the particle P at the instant t.

This establishes the Coriolis theorem.

Note on a generalized Ribaucour congruence

By K. N. KAMALAMMA, *Bangalore*

1. WE consider the congruence formed by lines through points on a surface S, parallel to the normals to another surface $\bar{S}$. If orthogonality of corresponding linear elements of S and $\bar{S}$ is assumed, the congruence is called a Ribaucour congruence.

S is called the surface of reference and $\bar{S}$ the director surface. The congruence is given by

$$\vec{R} = \vec{r} + t\vec{\bar{n}}, \tag{1}$$

where $\vec{r}$ is the position vector to any point on the surface of reference S, $\vec{\bar{n}}$ the unit normal vector at the corresponding point of the director surface $\bar{S}$ and t the distance along the ray measured from S.

2. If β represents the parameter of distribution for any ruled surface through a given ray, then β is given by

$$\beta = \frac{\left[\vec{r}', \vec{\bar{n}}', \vec{\bar{n}}\right]}{\vec{\bar{n}}'^2}$$

The explicit expression for β is

$$\begin{aligned}\beta = \{ & (\bar{L}\vec{r}_1 . \vec{\bar{r}}_2 - \bar{M}\vec{r}_1 . \vec{\bar{r}}_1)\, du^2 + (\bar{M}\vec{r}_1 . \vec{\bar{r}}_2 - \bar{N}\vec{r}_1 . \vec{\bar{r}}_2 + \\ & + \bar{L}\vec{r}_2 . \vec{\bar{r}}_2 - \bar{M}\vec{r}_2 . \vec{\bar{r}}_1)\, du\, dv + (\bar{M}\vec{r}_2 . \vec{\bar{r}}_2 - \bar{N}\vec{r}_2 . \vec{\bar{r}}_1)\, dv^2\} \times \\ & \times \{edu^2 + 2fdudv + gdv^2\}^{-1},\end{aligned} \tag{2}$$

where e, f, g are the first order magnitudes of the spherical representations of $\bar{S}$ and $\bar{L}$, $\bar{M}$, $\bar{N}$ are the magnitudes of the second order for $\bar{S}$.

Let the parametric curves on S correspond to the developables of the congruence. Hence

$$\bar{L}\vec{r}_1 . \vec{\bar{r}}_2 - \bar{M}\vec{r}_1 . \vec{\bar{r}}_1 = 0,$$

$$\bar{M}\vec{r}_2 . \vec{\bar{r}}_2 - \bar{N}\vec{r}_2 . \vec{\bar{r}}_1 = 0. \tag{3}$$

If the lines of curvature on $\bar{S}$ correspond to these developables, we have from the equations (3), since $\bar{M} = 0$, $\vec{r}_1 . \vec{\bar{r}}_2 = 0$. Hence $b = \vec{r}_2 . \vec{\bar{n}} = 0$; similarly $b' = \vec{r}_1 . \vec{\bar{n}}_2 = 0$. Thus the congruence is normal. Hence we have

THEOREM 1. *In the generalized congruence of Ribaucour, if the parametric curves on S correspond to the developables and if the lines of curvature on $\bar{S}$ correspond to these developables, then the congruence is normal.*

3. Following the notations of Weatherburn, the parameter of distribution β is given by

$$\frac{(af - b'e)\,du^2 + (ag - b'f + bf - ce)\,du\,dv + (bg - cf)\,dv^2}{h(e\,du^2 + 2f\,du\,dv + g\,dv^2)}.$$

The parametric curves correspond to the developables of the congruence if $af - b'e = 0$, $bg - cf = 0$. Hence

$$b + b' = \frac{f(ag + ce)}{eg}. \tag{4}$$

Taking the middle surface of the congruence as S, we have

$$f(b + b') - ag - ce = 0. \tag{5}$$

By (4) and (5) we get $ag + ce = 0$, $b + b' = 0$;

$$\beta = \frac{2(ag + bf)\,du\,dv}{h(e\,du^2 + 2f\,du\,dv + g\,dv^2)}.$$

The mean ruled surfaces correspond to the maximum and minimum values of β. These are given by

$$e\,du^2 - g\,dv^2 = 0.$$

If the null lines on $\bar{S}$ correspond to the developables, the equation to the mean ruled surfaces reduces to

$$\bar{L}\,du^2 - \bar{N}\,dv^2 = 0,$$

which is the differential equation of the lines of curvature on $\bar{S}$. Hence we have

THEOREM 2. *The mean ruled surfaces of the generalized congruence of Ribaucour meet the director surface $\bar{S}$ along its lines of curvature.*

This theorem has been proved for the strict Ribaucour congruence, i.e. for the congruence with orthogonality of corresponding linear elements, by R. S. Mishra and Srikrishna [*Ganita*, IV, 1, 1953].

4. Let the parametric curves on the surface of reference correspond as before to the developables. If now the asymptotic lines on S correspond to the developables, we have $\bar{L} = 0 = \bar{N}$ and hence from (3)

$$\vec{r}_1, \vec{\bar{r}}_1 = 0, \quad \vec{r}_2, \vec{\bar{r}}_2 = 0.$$

The two surfaces S and $\bar{S}$ therefore correspond with orthogonality of corresponding linear elements. Therefore the generalized congruence of Ribaucour reduces to the Ribaucour congruence. This provides a converse to the well-known result [Eisenhart, *Differential Geometry*, p. 421] that in a Ribaucour congruence the asymptotic lines correspond to the developables. Hence we have

THEOREM 3. *A necessary and sufficient condition that the generalized congruence considered in §1 is a Ribaucour congruence is that the asymptotic lines on S correspond to the developables.*

5. Let us now suppose that S and $\bar{S}$ are mutually inverse surfaces with orthogonality of the corresponding linear elements.

The mean ruled surfaces of the congruence of Ribaucour meet $\bar{S}$ in its lines of curvature and the mean ruled surfaces of the reciprocal congruence of Ribaucour [R. S. Mishra and Srikrishna, *ibid*] meet S in its lines of curvature. But it is well known that when a surface is inverted, lines of curvature invert into lines of curvature. Hence, we have

THEOREM 4. *If in a Ribaucour congruence the surface of reference and the director surface are inverse surfaces of each other, then the congruence and its reciprocal congruence have the same set of mean ruled surfaces.*

On the arithmetic-geometric mean inequality

By RICHARD BELLMAN, *Santa Monica, California*

THE most well-known inequality, which serves as the progenitor of all other general inequalities, is the inequality connecting the arithmetic and geometric means, namely

$$\frac{1}{N}\sum_{i=1}^{N} x_i \geqslant (x_1 x_2 \ldots x_N)^{1/n}, \tag{1}$$

for all $x_i \geqslant 0$. Equality occurs if and only if all the x_i are equal.

The object of this note is to present yet another proof of this inequality based upon the functional equation technique of the theory of dynamic programming [R. Bellman, *An introduction to the theory of dynamic programming*, Rand Report 245, 1953]. The basic idea is to regard the N-dimensional maximization as an N-stage process consisting of a sequence of one-dimensional maximizations.

To this end, let us set

$$f_N(a) = \max_{x} x_1 x_2 \ldots x_N, \tag{2}$$

where the variation is over the region defined by

$$x_i \geqslant 0, \ \sum_{i=1}^{N} x_i = a, \ a > 0. \tag{3}$$

Having chosen x_1, it is clear that the remaining variables x_2, $x_3, \ldots, x_N$ will be chosen so as to maximize the product $x_2 x_3 \ldots x_N$ subject to the constraint $\sum_{i=2}^{N} x_i = a - x_i$.

Hence, for $N = 2, 3, \ldots$, we have the recurrence relation

$$f_N(a) = \max_{0 \leqslant x_1 \leqslant a} x_1 f_{N-1}(a - x_1). \tag{4}$$

Since clearly $f_N(a) = a^N c_N$, where c_N is independent of a, we have

$$c_N = \max_{0 \leqslant x_1 \leqslant 1} x_1 (1 - x_1)^{N-1}. \tag{5}$$

The function $x_1(1-x_1)^{N-1}$ has a unique maximum at $x = 1/N$, for $N \geqslant 2$. Hence

$$c_N = c_{N-1} (N-1)^{N-1}/N^N. \tag{6}$$

Since $c_1 = 1$, we have $c_N = 1/N^N$. Furthermore, it follows that all the x_i are equal at the maximum point.

The same technique is applicable to many other problems possessing certain features of symmetry. It would, for example, seem to be the simplest way to determine the function

$$f_N(a, b) = \max_x \sum_{i=1}^N x_i, \tag{7}$$

where the variation is over the region described by

$$\text{(a)} \quad x_i \geqslant 0, \quad \text{(b)} \sum_{i=1}^N \phi(x_i) \leqslant a, \quad \text{(c)} \sum_{i=1}^N \psi(x_i) < b, \tag{8}$$

for general functions ϕ and ψ.

A note on matrix theory

By Richard Bellman, *Santa Monica, California*

The following proof of the well-known theorem that *the existence of a unique solution of $Ax = y$ for all y implies the same for the adjoint equation $A^T x = y$* may be of interest. Here x and y are N-dimensional vectors, A is an $N \times N$ matrix and A^T represents its transpose.

Consider the minimum over x of the quadratic form $(A^T x - y, A^T x - y)$, where (u, v) denotes the inner product, and let the minimum occur at $x = \bar{x}$. Then the minimizing condition is

$$(u, A(A^T \bar{x} - y)) = 0 \tag{1}$$

for all u, or $A(A^T \bar{x} - y) = 0$. Since, by assumption, *the* solution of $Az = 0$ is $z = 0$, we have $A^T \bar{x} - y = 0$, which shows that this equation has a solution.

To show that it is unique, let $A^T u = 0$, $u \neq 0$, and let v be the solution of $Av = u$, clearly non-zero. Then

$$0 < (u, u) = (Av, u) = (v, A^T u) = (v, 0) = 0, \tag{2}$$

a contradiction.

Under various assumptions, the above proof can be extended to cover more general functional equations.

Oblique co-ordinates

By Sahib Ram, *I. I. T., Kharagpur*

The object of this paper is to discuss the curvature of curves and find curves for which the tangent, subtangent, normal or subnormal is constant when the axes of reference are inclined at an angle ω.

I. CURVATURE.

1. Radius of curvature. Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be neighbouring points of a curve. The chord $PQ(\delta c)$ is evidently defined by the relation $\delta c^2 = (\delta x + \delta y \cos \omega)^2 + (\delta y \sin \omega)^2$. Assuming that

$$\lim_{Q \to P} \frac{\text{arc } PQ\ (\delta s)}{\text{chord } PQ\ (\delta c)} = 1,$$

we may now write

$$\left(\frac{ds}{dx}\right)^2 = (1 + y_1 \cos\omega)^2 + (y_1 \sin\omega)^2 = 1 + y_1^2 + 2\, y_1 \cos\omega, \quad (1)$$

$$\tan\psi = \frac{y_1 \sin\omega}{1 + y_1 \cos\omega}, \qquad (2)$$

where $y_1 = \dfrac{dy}{dx}$. Differentiating (2) we have

$$\sec^2\psi \frac{d\psi}{dx} = \frac{(1+y_1\cos\omega)\, y_2 \sin\omega - y_1 y_2 \sin\omega\cos\omega}{(1 + y_1\cos\omega)^2} = \frac{y_2 \sin\omega}{(1+y_1\cos\omega)^2}$$

which gives

$$\frac{d\psi}{dx} = \frac{y_2 \sin\omega}{(1 + y_1\cos\omega)^2\,(1 + \tan^2\psi)} = \frac{y_2\sin\omega}{(1 + 2y_1\cos\omega + y_1^2)}$$

Therefore the radius of curvature ρ is given by

$$\rho = \frac{d\psi}{ds} = \frac{(1 + 2y_1\cos\omega + y_1^2)^{3/2}}{y_2\sin\omega}.$$

2. Centre of curvature. The coordinates of $C(\xi, \eta)$, the centre of curvature at a point $P(x, y)$, are obtained, as given below, by using the sine formula for the triangle PCH where H is the meet of the lines through P and C parallel to the axes of x and y respectively:

$$\xi = x - \frac{P\cos(\omega - \psi)}{\sin\omega},$$

$$\eta = y + P\frac{\cos\psi}{\sin\omega}.$$

3. Conics. Coming to the well-known results for the ellipse, hyperbola and parabola, we get the same very quickly with the help of § 1.

(*i*) Consider the ellipse and hyperbola $\dfrac{x^2}{a'^2} \pm \dfrac{y^2}{b'^2} = 1$ referred to a pair of their respective conjugate diameters as axes, $2a'$, $2b'$ being their lengths; remembering that $a'b'\sin\omega = ab$, the radius

of curvature at an extremity $(0, b')$ of one of these axes is at once found to be a'^3/ab, y_1 being zero and $y_2 = \pm b'/a^2$, $2a$ and $2b$ being the lengths of major and minor or transverse and conjugate axes as the case may be.

(*ii*) Consider the parabola : $y^2 = 4a'x$ referred to a diameter and the tangent at its vertex where $a' = a \operatorname{cosec}^2 \delta$; the radius of curvature at the origin is easily seen to be $2a'^{3/2}/a^{1/2}$, $4a$ being the length of the latus-rectum and a' being the distance between the focus and the new origin.

(*iii*) Consider the parabola $\left(\frac{x}{a}\right)^{1/2} \pm \left(\frac{y}{b}\right)^{1/2} = 1$ referred to a pair of tangents. Differentiating we have

$$y_1 = \mp \left(\frac{by}{ax}\right)^{1/2} \text{ and } y_2 = \pm \frac{b}{2x(ax)^{1/2}}.$$

Substituting these values of y_1 and y_2 in the expressions for the coordinates of the centre of curvature at a point (x, y) we find the sum of the coordinates simplified to

$$x + y + \frac{1}{ab} [(ax)^{1/2} \pm (by)^{1/2}] [ax + 2 (abxy)^{1/2} \cos \omega + by] \sec^2 \omega/2$$

which turns to be independent of a and b and becomes equal to $3(x + y)$ when $a = b$ and $\omega = 90$ degrees—that means when the origin is at the meet of the axis and directrix of the parabola.

II. TANGENT AND NORMAL

4. Length of the tangent. The length of the tangent from its point $P(x, y)$ of contact with a curve to its meet with the x-axis is equal to

$$y \frac{\sin \omega}{\sin \psi} = \frac{\sqrt{1 + 2y_1 \cos \omega + y_1^2}}{y_1}.$$

If this is equal to a constant K then $K^2 y_1^2 = y^2(1 + 2y_1 \cos \omega + y_1^2)$ or $y_1^2(K^2 - y^2) - 2y^2 y_1 \cos \omega - y^2 = 0$, or

$$y_1 = \frac{y(y \cos \omega \pm \sqrt{K^2 - y^2 \sin^2 \omega})}{K^2 - y^2},$$

or

$$\frac{(K^2 - y^2)\, dy}{y(y \cos \omega \pm \sqrt{K^2 - y^2 \sin^2\omega})} = dx.$$

To solve this differential equation we put $y \sin \omega = K \sin \theta$, therefore $\sin \omega\, dy = \cos \theta\, d\theta$; the equation becomes

$$\begin{aligned} dx &= \frac{(\sin^2 \omega - \sin^2 \theta)\, K \cos \theta\, d\theta}{(\cos \omega \sin \theta \pm \sin \omega \cos \theta) \sin \theta \sin \omega} \\ &= \frac{K \sin(\omega + \theta) \sin(\omega - \theta) \cos \theta\, d\theta}{-\sin(\omega \pm \theta) \sin \theta \sin \omega} \\ &= \frac{-K \sin(\omega \mp \theta) \cos \theta\, d\theta}{\sin \omega \sin \theta} \\ &= -K \left[\frac{\cos^2 \theta}{\sin \theta} \mp \cot \omega \cos \theta \right] d\theta \\ &= -K [\operatorname{cosec} \theta - \sin \theta \mp \cot \omega \cos \theta]\, d\theta. \end{aligned}$$

$$x = C - K \left[\log \tan \frac{\theta}{2} + \cos \theta \mp \cot \omega \sin \theta\right],$$

$$y = K \sin \theta / \sin \omega,$$

which is easily recognised to be the tractrix when $\omega = 90°$.

5. Length of the normal. The length of the normal from a point $P(x, y)$ at which it is normal to a curve, to its meet with the x-axis is equal to

$$\frac{y \sin \omega}{\cos \psi} = \frac{y \sin \omega \sqrt{1 + 2y_1 \cos \omega + y_1^2}}{1 + y_1 \cos \omega}.$$

If this equals to a constant K, then

$$K^2(1 + y_1 \cos \omega)^2 = y^2 \sin^2 \omega (1 + 2y_1 \cos \omega + y_1^2)$$

or

$$y_1^2(K^2 \cos^2 \omega - y^2 \sin^2 \omega) + 2y_1 \cos \omega (K^2 - y^2 \sin^2 \omega) +$$
$$+ K^2 - y^2 \sin^2 \omega = 0,$$

or

$$y_1 = \frac{-(K^2 - y^2 \sin^2 \omega) \cos \omega \pm y \sin^2 \omega \sqrt{(K^2 - y^2 \sin^2 \omega)}}{(K^2 \cos^2 \omega - y^2 \sin^2 \omega)}$$

or

$$dx = \frac{K^2 \cos^2 \omega - y^2 \sin^2 \omega}{-(K^2 - y^2 \sin^2 \omega) \cos \omega \pm y \sin^2 \omega \sqrt{(K^2 - y^2 \sin^2 \omega)}} dy.$$

Solving this differential equation we get the curve as

$$x = C - \frac{K}{\sin \omega} \sin (\theta \pm \omega), \; y = \frac{K \sin \theta}{\sin \omega}.$$

Eliminating θ we obtain

$$(x + y \cos \omega - C)^2 = (K^2 - y^2 \sin^2 \omega),$$

or

$$x^2 + 2xy \cos \omega + y^2 - 2C (x + y \cos \omega) + C^2 - K^2 = 0$$

which is evidently a circle with radius equal to K.

Theory of functions of a real variable. By I. P. Natanson, translated into English from the original Russian by L. F. Boron and E. Hewitt, Frederick Ungar Publishing Co., New York.

This book, containing the translation of the first nine chapters of the original Russian book, presents in pleasant and easily readable language, an introductory course in the theory of Lebesgue integrals and Riemann-Stieltjes integrals of functions of one variable. The value of the book as a suitable text book for the student is enhanced by the translators' appendices to five out of nine chapters, extending the results in the text to cover more general situations. The chapter headings give a good idea of the contents of the book—infinite sets, point sets on the straight line, measurable sets, measurable functions, Lebesgue integrals of bounded functions, summable functions, square summable functions, functions of finite variation and Stieltjes integrals, absolutely continuous functions and the indefinite Lebesgue integrals. Useful exercises are given at the end of each chapter. The exposition is very lucid and pays great attention to details. This book is a valuable addition to the small number of text books on the topics covered by the book at this level.

V.G.

The metaphysics of logical positivism. By Gustav Bergmann, Longmans, Green and Co., 1954.

This book is not a treatise on logical positivism as the title might induce one to think. It consists of a selection of papers published by the author in various journals since 1946. All of them have been written for specialists, at any rate for scholars fairly well acquainted with various recent developments of symbolic logic and logical positivism. They call for much skill in the techniques of modern logic but also for an unlimited amount of patience when familiarizing oneself with an incredible number of delicate investigations about such simple sentences as " the rose is a flower "

We are told, at the very beginning of the first paper, that logical positivism " is a movement rather than a school ". It is not even a

label for a variety of schools of thought. However, this first paper gives an interesting account of the many currents (if not schools) of thought which have developed since 1930 around two centres, the continental one made famous by Ludwig Wittgenstein and Rudolf Carnap; the Cambridge centre, creation of G.E. Moore and Bertrand Russell. The author thinks it possible to find four common grounds of understanding between the leading logical positivists. First " all hold Humean views on causality and induction "; secondly they all insist " on the tautological nature of logical and mathematical truths "; thirdly they " conceive of philosophy as logical analysis, i.e. as a clarification of the language which we all speak in every day life "; lastly they all reject the possibility of metaphysics " in the sense that, e.g. the points at dispute among the traditional forms of idealisms, realism and phenomenalism could not even be stated, or at least could not be stated in their original intent in a properly clarified language ".

A reader who does not want, or cannot afford, to specialize in logical positivism will find it interesting to read the analysis of this movement given at the beginning of the second paper. In it, one finds a description of its main varieties, the definition of the " ideal language ", a well-knit criticism of Wittgenstein. This second paper is perhaps the most illuminating of the eighteen collected in the book under review, because it aims at clarifying and explaining the author's point of view, and his position called by him " reconstructionism ". However the reviewer feels obliged to confess that it is a much easier task to understand the author's objections to other systems than to acquire a clear idea of his own. In spite of two more papers in which this system is studied more elaborately, namely the sixth paper on " Bodies, Minds and Acts " and the last one on " Ideology ", one closes the book without being able to formulate in one's own familiar language the main theses which would contribute to some reconstruction of some kind of a metaphysics.

Yet it is the contention of the author, hinted at in the very title of the book and explicitly stated in his second paper " Logical Positivism, Language and Reconstruction of Metaphysics ", that it

is possible " to reconstruct in the new style the old Metaphysics ". In the new style, that is to say in the ideal language. There can hardly be any question of anything beyond a hope, at present, since the ideal language is still to be created. Or perhaps would it be more correct to speak of a faith in the possibility to beget an ideal language in a not too distant future? It looks extremely paradoxical that the logical positivist who wants to banish any faith from his intellectual life should cling so much to a faith, the faith in the possibility of a language whose very definition seems destined to arouse so many suspicions.

It would lead too far to analyze and criticize the author's contention. Suffice it to say that, in the reviewer's opinion, the possibility to construct an absolutely clear and flawless language making it easy for anyone to convey to others the meaning of his thoughts, or the contents of his observations, without any danger of obscurity or ambiguity, this very possibility seems to imply contradiction.

Naturally an ideal language expressing, by conventional words and symbols subjected to the rigid rules of a syntax things relative to a limited field of knowledge, for instance a game of chess, is not only an ideal but a reality. But as soon as one thinks of creating a conventional language whose field of applications should be coextensive with human knowledge, in particular with the First Philosophy, then one is bound to be defeated by the very nature of our knowledge and of its means of expression. Among many insuperable difficulties this one is most striking: a formalism intended to express restricted operations about a game of chess or an algebraic calculus can be introduced and explained in the language of common parlance. Now to extend the benefit of such an introduction to a language whose ambition it is to express the sum total of human knowledge leads to psychological and logical problems which cannot be deemed solvable. Unless the ideal language can be introduced and explained in this same language it is not an ideal language.

A remark of N. Bourbaki may well serve as a conclusion to this review. This eminent mathematician makes it very clear, in his " Theorie des Ensembles " —*Actualités Scientifiques et Industrielles*

No. 1212, *Hermann et Cie Editeurs, Paris* 1954—that if a conventional language could be constructed the simplest demonstrations of the theory of sets would require a few hundred symbols to be completely formalized. Further it may be safely assumed that the demonstrations themselves would cover a few hundred pages! Even in his systematic treatment of the general theory of sets, N. Bourbaki rules out the feasibility of a formalized text. It is not a practicable proposition. "Formalized Mathematics," he says "cannot be entirely written and it is necessary to trust constantly the common sense of the mathematician"

In spite of the criticism sketched, rather than explicitly stated, in this review of G. Bergmann's book, it would be wrong to conclude that the reviewer thinks such attempts as are made by the logical positivists futile or useless. The unremitting work of conscientious thinkers can never mean a negligible contribution to the progress of our knowledge. Even if they are ultimately proved to be unsolvable, the problems which these thinkers set and earnestly try to study always help us to make Philosophy, in spite of Wittgenstein's pessimistic phrase, a less and less "futile attempt to talk about the ineffable"

C. RACINE

The theory of numbers. By B. W. Jones, Constable & Co. (Orient Longmans Ltd., Bombay), pp. xi + 143.

This book deals with elementary number theory. No algebraic background is assumed, and the book can be read by students at the pre-university stage. The author has given the practical motivations for the problems considered in the book wherever possible. Some of the important theorems are put in as exercises, and the author expects the reader to anticipate the results. The exercises seem to be well chosen. The book is well written and the printing is nice, except for a few minor misprints.

In the first chapter integers are introduced by means of the Peano axioms and before defining rational numbers, the fundamental theorem of arithmetic and the Euclidean algorithm are

established and the decimal notation introduced. After the introduction of rational numbers, real numbers are defined by Dedekind sections and the chapter closes with the introduction of complex numbers. Chapter II deals with the properties of repeating decimals and the decimal notation is shown to lead in a natural way to the study of congruences. The usual properties of congruences and the allied theorems, including the Chinese remainder theorem, are proved. This chapter includes a discussion of the Fermat and Mersenne numbers. Chapter III deals with the linear and quadratic diophantine equations. Chapter IV begins with the study of simple continued fractions through Fibonacci sequences and concludes with the solution of the Pell's equation. Chapter V deals with non-linear congruences. The remainder theorem for polynomials modulo any integer is obtained and the chapter closes with a study of primitive roots. In the concluding chapter the topic is quadratic residues—Eisenstein's proof of the reciprocity law, properties of the Jacobi symbol and a short discussion on the sums of two squares.

V. Venugopal Rao

Introduction to factor analysis. By Benjamin Fruchter, D. Van Nostrand Company Inc., New York, pp. xii + 280.

In assessing the merits of any book it is essential to know to whom it is addressed in particular. In the preface to the book under review Prof. Fruchter hopes that " the book will serve as an introduction to the subject and as a stepping-stone to the more advanced texts ". According to the author, his main endeavour in producing this book is to introduce and develop the concept of factor analysis in logical order. The reader of the book is expected to know high school algebra with some elementary knowledge of analytical geometry and statistics. On reading this book, one feels that the author has done well within the bounds he has set for himself.

Factor-analytic procedures have developed considerably and there is a real need for such books as would bring the modern

techniques within the reach of the research workers in psychology and other fields, who lack the expert touch in factorial procedures owing to the paucity of their mathematical knowledge. Advanced text books of the type due to Thurstone, Holzinger and Hormon, Thompson and others are " out of bounds " for otherwise competent students because of the mathematics involved. Prof. Fruchter's book is evidently meant for this class. This class requires that the basic assumptions be made clear to them with an exposition of the computations involved in the analysis. The step-by-step explanation given in the book should meet this requirement adequately.

There are in all eleven chapters followed by an appendix and an extensive bibliography containing about 750 references. In the first two chapters the author introduces the reader to Spearman's two-factor theory, Holzinger's bi-factor solution and cluster analysis. Chapter 3 gives some matrix theory and ideas about vectors and their rotation. Though for a mathematical statistician it makes dull reading, the exposition should be welcome to the beginner. The treatment in Chapter 4, " Basic assumptions of Factor Analysis," leaves much to be desired. Factor analysis assumes some structure for the abilities or other characteristics which are latent either by their very nature or by the choice of the experimenter. It is the work of a factor analyst to determine this underlying structure. Spearman in his original contribution made certain hypotheses concerning the relationship of latent and observable variables. These hypotheses were later generalized in the works of those who followed. These hypotheses and their implications could have been discussed with no more mathematics than is already used in the book. The space (7 pages) allotted to this chapter is disproportionately small as compared to most of the other topics.

Chapter 5 deals with the computational aspects of diagonal and centroid methods of factoring. The author has illustrated the procedure with the help of an eleven variable example based on U.S. Air Force tests. The same example has been used to provide illustrations for other topics also. The choice of this example is to

be commended. While the multiple group and principal axes methods of factoring are dealt with in Chapter 6, the problem of rotation for orthogonal and oblique axes has been treated in Chapters 7 and 8 respectively. Some hints about interpretation of factors and report-writing have been given in Chapter 9.

The statistician who lacks expert knowledge in psychology can read this book without too much concern about the implications of the underlying model. He can borrow usefully many of the procedures for use in other fields. But it must be remembered, as pointed out by Cattel, that "factor analysis requires the skill of an art as well as of statistical principles". The author has not failed to point out this aspect of the technique. It is really a pleasure to go through Chapter 10, "Applications in Literature". In the ten examples that the author has selected one finds almost all the salient features of factoring. The examples are taken from such diverse fields as "supreme court votings in U.S." and "learning dynamics of bright and dull rats". These examples should go a long way to help understanding factor analysis procedures.

In a book of this type it would not have been out of order to indicate some sampling error formulae. After all, the entire structure of factor analysis is based on sampling and experimentation. Such a topic would have added greatly to the value of the book.

A minor slip may be pointed out. The author has introduced the term "standard score" on page 44 without defining it.

On the whole, Professor Fruchter's book is a valuable contribution to the existing literature on Factor Analysis. The reviewer feels that the book should prove useful as a text book for the beginner, and that in virtue of the exhaustive bibliography it would be an asset for the research worker.

V. S. Huzurbazar

NEWS AND NOTICES

The following persons have been admitted to life-membership of the Society: V. Ganapathy Iyer, V. S. Huzurbazar, R. Sankar, K. Satyanârayana.

The following persons have been admitted to membership in the Society: I. B. Bhaskara Rao, A. M. Chak, B. C. Das, K. M. Das, S. K. Deshpande, S. I. Husain, L. S. Kamat, S. K. Kaul, C. R. Marathe, N. L. Maria, Mahender Pratap, P. Rajagopal, N. Ramabhadran, D. L. Sharma, K. C. Sharma, Kirtan Singh, B. S. R. Somayajulu.

The following members of the American Mathematical Society have been admitted as members of the Society under the reciprocity agreement: N. J. Fine, F. W. Hoagland, J. A. Kalman, James D. Riley.

We regret to announce the death of the following members of the Society: H. F. Baker, S. Krishnamurti Rao, E. T. Whittaker.

Miss N. Padma and Miss K. Padmavalli have proceeded to the Universities of Chicago and North Carolina respectively for higher mathematical research.

Professor K. Chandrasekharan was on deputation in Europe from April 25 to June 9, 1956 to attend the meetings of the Executive Committee of the International Mathematical Union at Paris. He also gave lectures in the Universities of Rome, Paris and Göttingen.

Dr. P. L. Bhatnagar of the University of Delhi has been appointed Professor of Applied Mathematics at the Indian Institute of Science, Bangalore.

Sir R. P. Paranjpye has been appointed Vice-chancellor of the University of Poona.

The Second Congress on Theoretical and Applied Mechanics was held under the auspices of the Council of Scientific and Industrial

Research on October 15-16, 1956 at the National Physical Laboratory, New Delhi, under the presidentship of Professor K. S. Krishnan. About eighty delegates attended. The Congress received eighty-six communications — relating to finite deformation, elasticity theory, vibration and stability, fluid-flow, heat-transfer, ballistics and statistics—from many parts of the world including China, Czechoslovakia, Egypt, Japan, Poland, U.S.A. and U.S.S.R. Detailed information regarding the proceedings of the Congress may be had from Professor B. R. Seth, Indian Institute of Technology, Kharagpur.

The following office bearers of the Indian Society for Theoretical and Applied Mechanics were elected: S. R. Sen Gupta, President; V. Cadambe, N. R. Sen, Vice-Presidents; B. R. Seth, Secretary-Treasurer; Ram Ballabh, B. M. Belgaumkar, G. P. Chatterjee, A. K. Gayen, C. V. Joga Rao, S. Krishnan, V. Lakshimanarayanan, K. L. Rao, S. K. Roy, Members of the Council.

The Third Congress of the Society will be held in October-November 1957 at the Indian Institute of Science, Bangalore.

T. VIJAYARAGHAVAN

By K. CHANDRASEKHARAN

TIRUKKANNAPURAM Vijayaraghavan was born on November 30, 1902 in Adoor Agaram, a village in the South Arcot district of the State of Madras. He died of a heart attack on April 20, 1955 at Madras. His father was a distinguished scholar, both in Sanskrit and in Tamil, whose example left an abiding impression on the son's personality. His mother, who is a kind and gracious lady, and who has had to witness the tragedy of her son's sudden demise, is the centre of affection and reverence of the whole Vijayaraghavan family. From his parents Vijayaraghavan derived a passion for intellectual pursuits, for the study of literature and philosophy, ethics and religion. Interesting as those studies were, it was mathematics that presented a real challenge to his powers, and revealed the full scope of his genius.

He went to various schools, at Tirukkannapuram, Mannargudi, and Kudavasal, and matriculated from the Hindu Theological High School at Madras. He fared well at school, and won a few prizes. He did his Intermediate (1918-20) in Pachaiyappa's College, and his Honours (1920-24) in Presidency College, Madras. Judged by routine criteria, he did not do very well at college, and did not get the Honours. This is not altogether surprising, since he was genuinely interested, even at that stage, in serious mathematics, and had already begun thinking about research problems which could have stumped many of his teachers. To any one like him who was ready to set off on the path of mathematical discovery, much of the college curriculum and most of the teaching could hardly have seemed anything but boring. There was a shining exception, however, in Professor K. Ananda-Rau whose recognition of the young Vijayaraghavan's talent had not only been necessary for his admission to Presidency College, but sufficient to protect him from the grind of rules and regulations. In later years Vijayaraghavan used to recall with nostalgic pleasure his meetings, as a student,

with Ananda-Rau, and the inspiration that he had drawn from the teacher.

Even before he finished college, he published a research note (1920) on the set of limit points of the set $\{\phi(n)/n\}$, where $\phi(n)$ is the number of integers less than n and prime to n. He showed that every point in the interval (0, 1) is a limit point of the set $\{\phi(n)/n\}$. The problem is simple, and the solution not difficult. But the result is interesting; it is neither a routine generalization nor an obvious analogue of a known theorem. This aspect of his work is perhaps the most important. As he said once, he early began "to distinguish between research and research"

At about this time (1921—), he began sending some of his manuscripts to Professor G. H. Hardy in England. This must have seemed the obvious thing for him to do in view of the already well-known Ramanujan story. Hardy's response was not immediate. It was only some time later, after Mr. S. R. Ranganathan who had been a member of the mathematics staff of Presidency College had met Hardy and inquired about the matter, that Hardy had the manuscripts looked into, and wrote to the University of Madras that Vijayaraghavan's mathematical talent deserved recognition, and that he should be enabled to go to England to work with Hardy. The University of Madras accordingly offered him a scholarship to go to New College, Oxford (1925). This story, if not quite so romantic as Ramanujan's, has clearly a touch of similarity, since Vijayaraghavan had at that time no academic degree, and it was not only excusable but quite understandable if several people in Madras came to look upon him as a kind of spiritual successor to Ramanujan. It is not clear whether Vijayaraghavan himself felt that way, but there is no doubt that he was in the grip of the romance of Ramanujan, to the extent of believing that native wit rather than training and knowledge—to use his own words—was the principal requisite for research. Almost everyone would agree with that standpoint, but his singularity lay in the emphasis that he placed on the first at the expense of the second. It may seem somewhat paradoxical, though, in fact, it is revealing, that he placed

great emphasis on scholarship, comment, and criticism, as far as (Sanskrit or Tamil) literature was concerned—he was not creative in that field—, whereas in mathematics he paid a little less respect to such things, and reserved his admiration almost exclusively, and of course rightly, for original and creative work. Vijayaraghavan himself used to say, especially in his later years, that he liked people to tell him " what was happening ", and to come up to him with problems, so that he could have a go at them. His note on linearly ordered spaces is an instance in point. It was in the course of a casual conversation, on the work of a student supplicating for the M.Sc. degree, that Vijayaraghavan came across the two questions: Is a connected, linearly ordered topological space, which has the power of the continuum, separable? Is a connected topological space, of which every point is a cut point (i.e. its removal splits up the space into two and only two disjoint connected open sets), linearly ordered? Although his knowledge of set topology, at that time, hardly extended beyond the definitions, he produced the answers (which are in the negative in both cases), after three days or so of trial and error. As he said, this only showed that the problems were not deep, and that he could learn quickly; but it also showed, what Vijayaraghavan might not have been too ready to own, that to ask the 'right' questions was itself part of the game, and quite an interesting part.

Vijayaraghavan spent three years at Oxford (1925-28) working with Hardy. During that time he published several good papers, on divergent series, continued fractions and diophantine approximation; and, of course, got his D. Phil. In his brilliant preface to Hardy's *Divergent Series* Littlewood has remarked: "... in the early years of the century the subject, while in no way mystical or unrigorous, *was* regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity." Although it is difficult to decide whether Littlewood's "early years of the century" included the twenties, it seems quite proper to say that the subject of divergent series did offer, at the time Vijayaraghavan was working on it, problems which were neither

colourless nor dull, and his work did show the analytical finesse of a good and gifted mathematician.

He worked on Tauberian theorems, of the pre-Wienerian variety, and his first contribution (1926) consisted in a simplified proof of R. Schmidt's theorem on Abel means. The well-known converse of Abel's theorem, due to Tauber (1897), states that if Σa_n is Abel summable to a finite sum, and $na_n = o(1)$, then Σa_n converges. Littlewood (1910) replaced the condition $na_n = o(1)$ in Tauber's theorem by the much more general one $na_n = O(1)$; and Hardy and Littlewood (1914) replaced Littlewood's two-sided condition by the one-sided condition $na_n > -c$. R. Schmidt (1925) proved the result with the far more general condition

$$\liminf (s_m - s_n) \geqslant 0,\ s_m = \sum_0^m a_r,\ m > n,\ m \to \infty,\ n \to \infty,\ \frac{m}{n} \to 1.$$

Vijayaraghavan gave a simplified proof of Schmidt's theorem, without using the theory of the Stieltjes moment problem as Schmidt had done. Vijayaraghavan also extended (1928) his argument from Abel's method to Borel's method, and gave a simplified proof of another theorem of R. Schmidt: If Σa_n is summable by Borel's method to a finite sum s, and $s_n = \sum_{r=0}^{n} a_r$, and $\liminf\ (s_m - s_n) \geqslant 0$, $m > n$, $m \to \infty$, $n \to \infty$, $n^{-1/2}(m - n) \to 0$, then $s_n \to s$. Here again he dispensed with the use of the theory of the moment problems of Stieltjes and Hamburger, and applied, instead, an extension of the principle of repeated differentiation originally invented by Littlewood. All this work is interesting, and shows very intricate and deep analysis, but the subject was raised to an altogether different level by Norbert Wiener (1930) with the aid of the theory of Fourier transforms. However, in order to obtain the classical results as corollaries, Wiener did have to adopt methods which were strongly influenced by the work of his predecessors.

In the same line of research, he obtained (1927) converse theorems on summability (of series with real terms) when the sums are infinite, instead of being finite as usual. He showed that there is a

difference in behaviour between $(C, 1)$ means and Abel means on the one hand, and a difference in the operation of unilateral and bilateral Tauberian conditions on the other. He proved the following results:

1. If Σa_n is summable (A) to sum s, $|s| = \infty$, and $|a_n| < k/n$, then $\Sigma a_n = s$.
2. If Σa_n is summable (A) to sum s, $s = -\infty$, and $a_n > -k\,(n \log \log n)^{-1}$, then $\Sigma a_n = s$.
3. There is no sequence of negative numbers b_n such that Σb_n is divergent, for which the hypotheses that Σa_n is summable (A) to sum s, $s = \infty$, and $a_n \geqslant b_n$ imply that $\Sigma a_n = \infty$.
4. If Σa_n is summable $(C, 1)$ to sum s, $s = \infty$ and $a_n > -k\,n^{-1}$, then $\Sigma a_n = s$.
5. If Σa_n is summable $(C, 1)$ to sum s, $s = -\infty$, and $a_n > -k(n \log \log n)^{-1}$, then $\Sigma a_n = s$.

From (4) and (5) we see that in the case of $(C, 1)$ summability the Tauberian condition required when $s = +\infty$ is less stringent than when $s = -\infty$. In the case of (A) summability, on the other hand, we observe from (2) and (3) that if $s = +\infty$, no strengthening of the Tauberian condition will yield a positive, non-trivial result, whereas if $s = -\infty$, the result is analogous to the $(C, 1)$ case. We also observe from (1) that these niceties are absent in the case of a two-sided Tauberian condition.

In connexion with Vijayaraghavan's Tauberian theorems should also be mentioned his Mercerian theorem. Mercer proved (1907) that if

$$A_n \equiv s_n + a\,t_n \equiv s_n + a\,\frac{s_1 + \dots + s_n}{n} \to s, \text{ as } n \to \infty,$$

where $a > 1$, then $s_n \to s$. Between 1912 and 1923 there appeared a variety of proofs of this theorem, including one by Hardy and another by Knopp. Vijayaraghavan generalized it (1928) by proving that if $s_n + q_n t_n \to 0$, with $\liminf q_n > -1$, then $t_n \to 0$. If the additional condition that q_n is bounded is imposed, then $s_n \to 0$, thereby yielding Mercer's theorem as a corollary.

Besides Tauberian theorems, Vijayaraghavan published (1927) a paper on continued fractions, a subject in which he began to evince interest since his Honours days. In proving the classical result of Lagrange that every quadratic surd corresponds to a periodic, simple, continued fraction, it is shown that if a quadratic surd has the representation $(P + \sqrt{R})/Q$, where $|P|$, $|Q|$, R are (the least) positive integers for which it is true that $Q \mid (R - P^2)$, then the number of elements in the periodic part of the corresponding simple continued fraction is less than $2R$. Vijayaraghavan improved this estimate by showing, in an elementary way, that the $2R$ can be replaced by $O(R^{\frac{1}{2}+\epsilon})$, $\epsilon > 0$. More precisely, he showed, by using some results of I. Schur, that if $N(R) \equiv N(P, Q, R)$ denotes the number of elements in the smallest period of the continued fraction corresponding to $(P + \sqrt{R})/Q$, then $N(R) = O(R^{\frac{1}{2}} \log R)$, and that for an infinity of values of R, $N(R) > R^{\frac{1}{2}-\epsilon}$, $\epsilon > 0$, and for an infinity of R, $N(R) = O(\log R)$, the O's being uniform with respect to P and Q. Associated with this result is another by him on the solvability of the Pellian equation $Rx^2 = y^2 + 1$. It is well known that this equation is, or is not, solvable in integers according as the continued fraction for $\sqrt{R}$ has, or has not, a period with an odd number of elements. Vijayaraghavan considered the more general surd $(P + \sqrt{R})/Q$, and showed that it has, or has not, a period with an odd number of elements, according as the Pellian equation is, or is not, solvable for R.

Vijayaraghavan's paper (1927) on Diophantine approximation arose out of the famous memoir on the subject by Hardy and Littlewood [*Acta Math.* 37 (1914), 155-190]. Let (x) denote the fractional part of x, and $\phi \equiv \phi(\lambda, \theta, \alpha)$ denote the least positive integer n. for which the approximation $|(n\theta) - \alpha| < 1/\lambda$ holds. Confirming a conjecture of Hardy and Littlewood, Vijayaraghavan showed that for particular choices of θ and α one could make ϕ arbitrarily large, for a sequence of values of λ tending to infinity. More precisely, given $\Omega \equiv \Omega(\theta, \alpha)$ and $\psi = \psi(\lambda, \alpha)$, which are two arbitrary functions, finite for all values of their arguments, the first independent of λ, and the second of θ, one can find an α for which

$0 \leqslant \alpha < 1$, and an irrational number θ, such that $\phi > \Omega \psi$, for an infinity of values of λ. Since Kronecker's fundamental theorem on diophantine approximation suggests the plausibility of a similar result with any finite number of θ's, Vijayaraghavan considered the case of two θ's, and proved something different, namely: If $\phi \equiv \phi(\lambda, \theta_1, \theta_2, \alpha_1, \alpha_2)$ denotes the least positive integer n for which it is simultaneously true that

$$|(n\,\theta_1) - \alpha_1| < \frac{1}{\lambda},\ |(n\,\theta_2) - \alpha_2| < \frac{1}{\lambda},$$

then given $\Omega \equiv \Omega(\theta_1, \theta_2, \alpha_1, \alpha_2)$, $\psi \equiv \psi(\lambda, \alpha_1, \alpha_2)$, which are two arbitrary functions, one can find two linearly independent irrationals θ_1 and θ_2, and two numbers α_1 and α_2, such that for *all* large λ, $\phi > \Omega \psi$. It is to be noted that whereas in the first case, the result on the increase of ϕ holds for a sequence of values of λ, in the second case the result holds for *all* large λ. This difference is interesting because it results in different answers, in the two cases, to the question of how small, instead of how large, one can make ϕ. It had been observed by Hardy and Littlewood, in the first case, that to any θ and α there corresponds a sequence of values of λ tending to infinity, for which $\phi < \frac{1}{2}\lambda$. Vijayaraghavan's result shows that in the case of two θ's there is no such theorem.

After his return home from Oxford (1928) Vijayaraghavan took up an appointment in Annamalai University for a year; he moved to Aligarh, as a Lecturer, in 1930. There he came in contact with Professor André Weil for whom he conceived a great admiration. In 1931 he went to Dacca University, as a Reader, and remained there till 1946. The first important piece of work after his return from Oxford relates to a conjecture of Borel.

Borel proved in 1899 that if $y = f(x)$ is a real continuous function of a real variable x, which satisfies an algebraic differential equation of the first order, say

$$F(x, y, y') = 0,$$

for $x > x_0$, where y' is the first derivative of f, and F is a polynomial in x, y, y', then

$$y = o(e^{e^x}) \equiv o(e_2(x)).$$

Borel conjectured that similar results would hold for differential equations of higher order; in particular, for equations of the second order, he believed that he could prove that

$$y = o(e_3(x)).$$

Lindelöf [*Bull. Soc. Math.* 17 (1899), 205] gave a more precise form of Borel's theorem, namely that if equation (*) is of degree m in x, there exists a constant c, such that

$$y > e^{cx^{m+1}},$$

for $x > x_0$. Hardy took up this problem in 1912 [*Proc. London Math. Soc.* 10 (1912) 451-468], and remarked that, in the case of the second-order equation, Borel's proof is "not complete, but its general lines are indicated". He added : "There is no doubt of the truth of the corresponding general theorem, though, so far as I am aware, no strict proof has ever been given". In his paper Hardy did not attack the general problem, but obtained refinements of Borel's theorem in the case of first-order equations. Any one who looked into Hardy's paper would have felt the urge to prove the plausible general theorem. Vijayaraghavan showed, however, that the analogue of Borel's theorem, for equations of higher order, was false; and that, even in the case of first order equations, it would be false if the restriction that the solution should be real was not imposed. His proof (1932) is ingenious, simple, and beautiful, and shows Vijayaraghavan at his best. It consists in considering the real part of a Weierstrassian elliptic function with suitably chosen periods, and deservedly attracted the notice of distinguished mathematicians like G. D. Birkhoff. It was only a few years later (1936) that, on Birkhoff's initiative, he was invited to the United States as a Visiting Lecturer of the American Mathematical Society. Vijayaraghavan used to cherish the memories of that visit, and long before American universities earned "recognition" in Indian bureaucratic circles, he used to impress on his younger colleagues, some of the great things for which American universities stood.

Vijayaraghavan returned in 1937 to his disproof of Borel's conjecture, and in collaboration with his colleague at Dacca, Professor N. M. Basu, and with the distinguished physicist Professor S. N. Bose, constructed an extremely simple example: If $\phi(x)$ is an arbitrary, real, increasing function, tending to $+\infty$ with x, and

$$f(x) = \frac{1}{2 - \cos x - \cos \alpha x},$$

for a suitably chosen irrational α, depending on ϕ, then

$$|f(x)| > \phi(x)$$

for a sequence of values of x tending to infinity, although f is a real, continuous solution of an algebraic differential equation of the second order. Actually α is chosen in the following way. Let $\{d_n\}$ be a sequence of positive integers, such that

$$d_r > 4\pi\, \phi(2\pi\, q_{r-1}),\ r = 2, 3, 4, \ldots,$$

where $q_r = d_1 d_2 \ldots d_r$, $r = 1, 2, 3 \ldots$. Then take $\alpha = \sum_{r=1}^{\infty} 1/q_r$.

While the simplicity of this example should not moderate one's praise for Vijayaraghavan's ingenuity in the first place, it does remind us that the value of "negative" results is a changeable thing, and that in a total assessment of one's work, as distinct from one's powers, negative results occupy a relatively minor place.

His next major interest was the study of the fractional parts of powers of real numbers. Let θ be a real number. Consider θ^n, $n = 1, 2, \ldots$ Let $G(\theta)$ denote the set of limit points of the fractional parts of θ^n. It was proved by Hardy and Littlewood [*Acta Math.* 37 (1914), 181] in their study of some problems of Diophantine approximation that the set E of real numbers $\theta > 1$ for which $G(\theta)$ is not the entire unit interval is of Lebesgue measure zero. The problem of determining the arithmetical nature of the 'exceptional' set E naturally arose. Hardy himself took this up in 1919 [*J. Indian Math. Soc.*, 11 (1919), 162-166], and quickly realized that while the general problem was "one of great difficulty", the answer was

"almost immediate" in the case in which θ was an algebraic integer. He proved the following theorem: Let (x) denote the fractional part of x. Let a be a real, algebraic number, greater than 1, which is the root of an irreducible equation

$$k_0 a^m + k_1 a^{m-1} + \dots + k_m = 0,$$

where $k_0, k_1, \dots, k_m$ are integers. Then in order that numbers $\alpha > 0$ should exist which satisfy the condition

$$(\alpha a^n) \to 0, \text{ as } n \to \infty,$$

it is necessary and sufficient that $k_0 = 1$, so that a is an algebraic integer, and that the moduli of all the roots of the above equation, other than a itself, should be less than 1.

In his series of four papers on the subject, Vijayaraghavan proved, among other things, the following theorem which generalizes Hardy's, in as much as it considers the case in which $G(\theta)$ consists of any finite number of points, instead of only one (namely the point zero) as in Hardy's case: If $\{u_n\}$ is a sequence of real numbers

$$u_n = d_1 \xi_1^n + d_2 \xi_2^n + \dots + d_l \xi_l^n, \quad n = 1, 2, 3, \dots,$$

where $d_1 . d_2 \dots d_l \neq 0$, $\xi_1, \xi_2, \dots, \xi_l$ are distinct algebraic numbers with $|\xi_r| > 1$, and if the limit points of fractional parts of u_n are finite in number, then $\xi_1, \xi_2, \dots, \xi_l$ are algebraic integers; moreover, if η is a conjugate of one or more of the numbers $\xi_1, \dots, \xi_l$, but is not one of them, then $|\eta| < 1$. This theorem led him to ask whether the hypothesis that $G(\theta)$ is a finite set implies, in itself, that θ is algebraic. Although he could not prove that, he showed that the set of θ's for which $G(\theta)$ is a finite set is enumerable, and the set of θ's for which $G(\theta)$ is not the entire unit interval (which had been shown to be of measure zero by Hardy and Littlewood) has the power of the continuum.

Vijayaraghavan was led to this general result by a series of simple but interesting results. Throughout these it turns out that the class S of algebraic integers θ whose conjugates, other than θ, have an

absolute value less than 1, plays a special part. This class, which has been sometimes referred to as the class of Pisot-Vijayaraghavan numbers (since M. Pisot had independently encountered it) originated explicitly in Hardy's paper (1919, loc. cit) on the same subject.

In attempting a study of $G(\theta)$, for real $\theta > 1$, Vijayaraghavan began with the simplest case in which θ is rational. In this case he proved that $G(\theta)$ has an infinity of points (Weil gave an alternative proof of this result). On the other hand, if $\theta \in S$, he showed that $G(\theta)$ has only a finite number of points. This, however, does not mean that for all algebraic integers θ, $G(\theta)$ is a finite set, for he showed that there exist algebraic integers θ for which $G(\theta)$ consists of the entire interval $(0, 1)$. It is still not known if every real algebraic $\xi \geqslant 1$ which is not in S is such a θ. It follows, as a particular case of his theorem quoted earlier, that for all real algebraic *numbers* $\xi \geqslant 1$ other than those that belong to S, $G(\xi)$ has an infinity of points.

The part played by the set S in the above considerations led Vijayaraghavan and others to study it more closely. Vijayaraghavan himself showed that S contains algebraic integers of all degrees, and gave some of the limit points of S which are also members of the set, and conjectured that it was improbable that S could be everywhere dense in some interval, or be dense in itself. Later Salem showed [*Duke Math. J.* 11 (1944), 103-108] that the set S is closed and since it is enumerable, it follows that it is nowhere dense, nor dense in itself, confirming Vijayaraghavan's conjecture. Siegel determined [*Duke Math. J.* 11 (1944), 597-602] the first two members of the set S to be θ_1 and θ_2, where θ_1 is the positive zero of $x^3 - x - 1$ and θ_2 the positive zero of $x^4 - x^3 - 1$. Siegel further showed that any other point of the set is greater than $\sqrt{2}$. The set is connected with the study of sets of uniqueness and of multiplicity for trigonometric series, through Salem's conjecture that the perfect set P, of Cantor's type, with the constant ratio of dissection $(\xi, 1 - 2\xi, \xi)$, $0 < \xi < \frac{1}{2}$, is a set of uniqueness for trigonometric series if and only if $1/\xi$ belongs to S. Salem has proved

[*Trans. American Math. Soc.* 54 (1943), 218-228; 56 (1944), 32-49; 63 (1948), 595-598] that if $1/\xi \notin S$, then P is a set of multiplicity, though it is not yet known whether P is a set of uniqueness if $1/\xi \in S$.

Although Vijayaraghavan wrote several other papers, none of which is trivial, it seems safe to say that his proof of the Tauberian theorem on Borel summability, his disproof of Borel's conjecture, his note on Diophantine approximation, and his work on the limit points of fractional parts of powers of a real number, constitute his best achievement. He was found, on at least one occasion, to express an opinion akin to that.

In 1946 he resigned his position at Dacca, and was appointed to a Professorship in Andhra University. That was the first chance he got, as he used to remark, of heading a department. He had plans of some sort for building up a school at Waltair, but they never came to fruition, and he left that university in 1949 to be Director of the privately endowed Ramanujan Institute of Mathematics, intended to be located at Karaikudi, but functioning, for the time being, at Madras. It was his ambition to make that Institute an important research centre, and he sought the support of the Government of India to that end. The response, one imagines, was neither immediate nor adequate, and he passed away before his dream could be realized.

Reviewing his work, one cannot fail to be impressed by its uniformly high quality. His analysis was in the genuine Hardy tradition. His whole approach to mathematics was strongly influenced by Hardy whom he admired not only as a mathematician but as a truly great individual. He liked to solve problems, and classified them according to their order of importance. He was a mathematician of taste, and of considerable strength; he never liked to publish anything trivial; he strove for originality. He was precise. On the other hand, all his main problems, almost entirely, arose out of the work of Hardy and Littlewood. He did no pioneering; he was interested in theorems, not theories. He viewed

mathematics, more or less, as an infinite, discrete set of interesting problems. The structures of mathematics, their inter-relationship and architecture were somewhat unreal to him. His mathematical powers were undoubtedly greater than his achievements. And no one who knew him intimately,—as a working mathematician, as a delightful conversationalist, as a genial host, or as an affectionate father,—could fail to say : here was an intellectual of whom his country could be proud.

Vijayaraghavan loved lecturing, and was a lucid, effective, and sometimes brilliant lecturer, especially on mathematical topics which were of immediate interest to him. His dialectical skill and sophistication were used to great advantage. He loved to give alternative proofs, counter-examples, plausible arguments, and generalizations; and it gladdened his heart to know that at least some members of the audience appreciated his effort and his skill. It was a pet saying of his that one could not claim that one knew a theorem, unless one could give not less than three different proofs of it, of which at least one proof was one's own. There was a mock-serious occasion when this saying landed him in considerable embarrassment. He was desirous of attracting good students, and believed, as he well might, that he did his very best to encourage talent. It has to be admitted, however, that his efforts were not rewarded with success, and he felt somewhat frustrated and disappointed. It is a surprise that such a truly accomplished lecturer attracted so few 'regular' pupils. At least a part of the answer is to be found in the intellectual milieu in which his lot was cast. Another point is that his intellect functioned according to Toynbee's principle (regarding civilizations) of challenge and response. He needed intellectual companionship of a kind provided by a Hardy or a Weil or a Pillai. This was vouchsafed to him only for brief intervals, and although his reputation rests on solid foundations, it could, and perhaps would, have been greater under more propitious circumstances.

Vijayaraghavan was for many years a member of the Indian Mathematical Society, of the London Mathematical Society and of

the American Mathematical Society. From 1947 to 1955 he functioned as a member of the Council of the Indian Mathematical Society, first as Secretary, then as President, and later as Librarian. He was ever in favour of establishing a high standard for the Society's publications, and gave his unreserved support for many proposals aimed at bringing it about. His premature death is mourned not only by his aged mother, his wife and children, but by the entire mathematical community in India and by his many distinguished friends abroad.

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Tata Institute of Fundamental Research
Bombay

The above list of publications by Vijayaraghavan, with the exception of two items, was kindly supplied to me by Prof. C. T. Rajagopal.